MIMO QFT Controller Design Reformulation – Application to Spacecraft with Flexible Appendages & Formation Flying in Low Earth Orbit

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ABSTRACT

This paper summarizes a new methodology to design sequential non-diagonal QFT controllers for multi-input-multi-output MIMO systems with uncertainty. It also demonstrates the feasibility of that methodology to control both, a) the position and attitude of a 6x6 MIMO spacecraft with flexible appendages, and b) spacecraft flying in formation in low Earth orbit.

1.0 INTRODUCTION

Control of multivariable systems (multiple-input-multiple-output, MIMO) with model uncertainty are still one of the hardest problems that control engineers have to face in Spacecraft real-world applications. Three of the main complexities that define a MIMO system are the input and output directionality -different vectors to actuate \( U \) and to measure \( Y \); the coupling among control loops -some outputs \( y \) can be influenced by several inputs \( u \), and some inputs \( u \) can influence several outputs \( y \); and the transmission zeros of the plant matrix.

2.0 NON-DIAGONAL MIMO QFT CONTROL DESIGN METHODOLOGY

Control of multivariable systems (multiple-input-multiple-output, MIMO) with model uncertainty are still one of the hardest problems that the control engineer has to face in real-world applications. Three of the main characteristics that define a MIMO system are the input and output directionality -different vectors to actuate \( U \) and to measure \( Y \); the coupling among control loops -some outputs \( y \) can be influenced by several inputs \( u \), and some inputs \( u \) can influence several outputs \( y \); and the transmission zeros of the plant matrix.
In the last few decades a very significant amount of work in MIMO systems, too numerous to list here, has been done. Using MIMO QFT, Horowitz proposed to translate the original $n \times n$ MIMO problem into $n$ separate quantitative multiple-input-single-output (MISO) problems, each with plant uncertainty, external disturbances and closed-loop tolerances derived from the original problem [1]. Two different approaches, the so-called sequential and non-sequential methods, consider in successive iterative steps an equivalent plant that either takes also into account the controllers designed in the previous steps, or only deals with the plant respectively.

However, although such original MIMO QFT methods take the coupling among loops into account, they only propose the use of a diagonal controller $G$ to govern the MIMO plant. This structure can be improved using non-diagonal controllers. In fact, a fully populated matrix controller allows the designer much more design flexibility to control MIMO plants than the classical diagonal controller structure. The use of the non-diagonal components can also ease the diagonal controller design problem. In the last few years some new methods for non-diagonal multivariable QFT robust control system design have been introduced. For the sake of clarity, this section summarizes a previous work [2-7] that extends the classical QFT diagonal controller design for MIMO plants with uncertainty to the fully populated matrix controller design. The work studies three cases: the reference tracking, the external disturbance rejection at plant input and the external disturbance rejection at plant output. It presents the definition of three specific coupling matrices ($c_{1ij}$, $c_{2ij}$, $c_{3ij}$), one for each case, and introduces a sequential design methodology for non-diagonal QFT controllers.

### 2.1 The Coupling Matrix

The objective of this section is to define a measurement index (the coupling matrix) that allows one to quantify the loop interaction in MIMO control systems. Consider a $n \times n$ linear multivariable system -see Fig. 1-, composed of a plant $P$, a fully populated matrix controller $G$, a pre-filter $F$, a plant input disturbance transfer function $P_{di}$, and a plant output disturbance transfer function $P_{do}$, where $P \in \mathbb{S}$, $\mathbb{S}$ is the set of possible plants due to uncertainty, and,

$$P(s) = \begin{bmatrix} p_{11}(s) & p_{12}(s) & \cdots & p_{1n}(s) \\ p_{21}(s) & p_{22}(s) & \cdots & p_{2n}(s) \\ \vdots & \vdots & \ddots & \vdots \\ p_{n1}(s) & p_{n2}(s) & \cdots & p_{nn}(s) \end{bmatrix} ; \quad G(s) = \begin{bmatrix} g_{11}(s) & g_{12}(s) & \cdots & g_{1n}(s) \\ g_{21}(s) & g_{22}(s) & \cdots & g_{2n}(s) \\ \vdots & \vdots & \ddots & \vdots \\ g_{n1}(s) & g_{n2}(s) & \cdots & g_{nn}(s) \end{bmatrix} ; \quad F(s) = \begin{bmatrix} f_{11}(s) & f_{12}(s) & \cdots & f_{1n}(s) \\ f_{21}(s) & f_{22}(s) & \cdots & f_{2n}(s) \\ \vdots & \vdots & \ddots & \vdots \\ f_{n1}(s) & f_{n2}(s) & \cdots & f_{nn}(s) \end{bmatrix}$$

(1)

The reference vector $r'$ and the external disturbance vectors at plant input $d_i'$ and plant output $d_o'$ are the inputs of the system. The output vector $y$ is the variable to be controlled.

It is denoted $P^*$ as the plant inverse so that,

$$P(s)^{-1} = P^*(s) = [p_{ij}^*(s)] = A(s) + B(s) = \begin{bmatrix} p_{11}^*(s) & 0 & 0 \\ 0 & \cdots & 0 \\ 0 & 0 & p_{nn}^*(s) \end{bmatrix} + \begin{bmatrix} 0 & \cdots & p_{1n}^*(s) \\ \vdots & \ddots & \vdots \\ p_{n1}(s) & \cdots & 0 \end{bmatrix}$$

(2)
\[ G(s) = G_d(s) + G_b(s) = \begin{bmatrix} g_{11}(s) & 0 & 0 \\ 0 & \ldots & 0 \\ 0 & 0 & g_{mm}(s) \end{bmatrix} + \begin{bmatrix} 0 & \ldots & 0 \\ 0 & \ldots & 0 \\ g_{ab}(s) & \ldots & 0 \end{bmatrix} \]  

(3)

where \( A \) is the diagonal part and \( B \) is the balance of \( P^* \); and \( G_d \) is the diagonal part and \( G_b \) is the balance of \( G \). The next paragraphs introduce a measurement index to quantify the loop interaction in the three classical cases: reference tracking, external disturbances at plant input, and external disturbances at plant output. That index is called the coupling matrix and, depending on the case, shows three different expressions: \( C_1, C_2, C_3 \) respectively.

**Fig. 1** Structure of a 2 Degree of Freedom MIMO System

### 2.1.1 Tracking

The transfer function matrix of the controlled system for the reference tracking problem, without any external disturbance, can be written as shown in Eq. (4),

\[ y = (I + P G)^{-1} P G r = T_{y/r} r = T_{y/r} F r' \]  

(4)

Using Eq. (2) and (3), Eq. (4) can be rewritten as,

\[ T_{y/r} r = (I + A^{-1} G_d)^{-1} A^{-1} G_d r + (I + A^{-1} G_d)^{-1} A^{-1} (G_b (r - (B + G_b) T_{y/r} r)) \]  

(5)

In the expression of the closed-loop transfer function matrix of Eq. (5), it is possible to find two different terms:

i. A diagonal term \( T_{y/r,d} \),

\[ T_{y/r,d} = (I + A^{-1} G_d)^{-1} A^{-1} G_d \]  

(6)

that presents a diagonal structure. Note that it does not depend on the non-diagonal part of the plant inverse \( B \), nor on the non-diagonal part of the controller \( G_b \). It is equivalent to \( n \) reference tracking SISO systems formed by plants equal to the elements of \( A^1 \) when the \( n \) corresponding parts of a diagonal \( G_d \) control them, as shown in Fig. 2a.

ii. A non-diagonal term \( T_{y/r,b} \),

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that presents a non-diagonal structure. It is equivalent to the same \( n \) previous systems with internal disturbances \( c_{ij} r_j \) at plant input (Fig. 2b).

In Eq. (7), the matrix \( C_1 \) is the only part that depends on the non-diagonal parts of both the plant inverse \( B \) and the controller \( G_b \). Hence, it comprises the coupling, and from now on \( C_1 \) will be the coupling matrix of the equivalent system for reference tracking problems,

\[
C_1 = G_b - (B + G_b) T_{y/r}
\]

Each element \( c_{ij} \) of this matrix obeys,

\[
c_{ij} = g_{ij} (1 - \delta_{ij}) - \sum_{k=1}^{n} \left( p_{ik} + g_{ik} \right) t_{kj} (1 - \delta_{ik})
\]

where \( \delta_{ki} \) is the delta of Kronecker that is defined as,

\[
\delta_{ki} = \begin{cases} 
1 & \text{if } k = i \\
0 & \text{if } k \neq i 
\end{cases}
\]

2.1.2 Disturbance rejection at plant input

The transfer matrix from the external disturbance at plant input \( d_i \) to the output \( y \) can be written as shown in Eq. (11),

\[
y = (I + P G)^{-1} P d_i = T_{y/di} d_i = T_{y/di} P_{di} d_i
\]
and then,

\[ T_{y/di} d_i = \left( I + A^{-1} G_d \right)^{-1} A^{-1} d_i - \left( I + A^{-1} G_d \right)^{-1} A^{-1} \left( B + G_b \right) T_{y/di} d_i \]  \tag{12}

In that expression -Eq. (12)- it is possible to find two different terms:

i. A diagonal term \( T_{y/di, d} \)

\[ T_{y/di, d} = \left( I + A^{-1} G_d \right)^{-1} A^{-1} \]  \tag{13}

Again, Eq. (13) is equivalent to \( n \) regulator MISO systems, as shown in Fig. 3a.

ii. Non diagonal term \( T_{y/di, b} \)

\[ T_{y/di, b} = \left( I + A^{-1} G_d \right)^{-1} A^{-1} \left( B + G_b \right) T_{y/di} = \left( I + A^{-1} G_d \right)^{-1} A^{-1} C_2 \]  \tag{14}

that presents a non-diagonal structure which is equivalent to the same \( n \) previous systems with external disturbances \( c_{2ij} \) \( di_j \) at plant input, as shown in Fig. 3b.

In Eq. (14), the matrix \( C_2 \) comprises the coupling, and from now on \( C_2 \) will be the coupling matrix of the equivalent system for external disturbance rejection at plant input problems,

\[ C_2 = \left( B + G_b \right) T_{y/di} \]  \tag{15}

![Fig. 3](image)

Each element \( c_{2ij} \) of this matrix obeys,
where $\delta_{ki}$ is the delta of Kronecker defined in Equation (10).

### 2.1.3 Disturbance rejection at plant output

The transfer matrix from the external disturbance at plant output $d'_o$ to the output $y$ can be written as shown in Eq. (17),

$$y = (I + PG)^{-1} d'_o = T_{y/od} d'_o = T_{y/do} P_{do} d'_o$$

and then,

$$T_{y/od} d'_o = (I + A^{-1} G_d)^{-1} d'_o + (I + A^{-1} G_d)^{-1} A^{-1} (B + G_b) T_{y/do} d'_o$$

In that expression -Eq. (18)- it is possible to find two different terms:

i. A diagonal term $T_{y/do,d}$,

$$T_{y/do,d} = (I + A^{-1} G_d)^{-1}$$

Once more, Eq. (19) is equivalent to the $n$ regulator MISO systems showed in Fig. 4a,

ii. Non diagonal term $T_{y/do,b}$

$$T_{y/do,b} = (I + A^1 G_d)^{-1} A^{-1} \left[ B + (B + G_b) T_{y/do} \right] = (I + A^{-1} G_d)^{-1} A^{-1} C_3$$

that presents a non-diagonal structure. It is equivalent to the same $n$ previous systems with external disturbances $c_{3ij} do_j$ at plant input, as shown Fig. 4b.

In Eq. (20), the matrix $C_3$ comprises the coupling, and from now on it will be the coupling matrix of the equivalent system for external disturbance rejection at plant output problems,

$$C_3 = B + (B + G_b) T_{y/do}$$

Each element of the coupling matrix, $c_{3ij}$ obeys,

$$c_{3ij} = p_{ij}^* (1 - \delta_{ij}) - \sum_{k=1}^{n} (p_{ik}^* + g_{ik}) t_{kj} (1 - \delta_{ki})$$

where $\delta_{ki}$ is the delta of Kronecker as defined in Equation (10).
2.2 The Coupling Elements

In order to design a MIMO controller with a low coupling level, it is necessary to study the influence of every non-diagonal element $g_{ij}$ on the coupling elements $c_{1ij}$, $c_{2ij}$ and $c_{3ij}$, defined in Eq. (9), (16) and (22). These elements can be simplified to quantify the coupling effects. Then it will be possible to analyze the loop decoupling and to state some conditions and limitations using fully populated matrix controllers. To analyze the coupling elements, one Hypothesis is stated.

**Hypothesis H1**: suppose that in Eq. (9), (16) and (22),

$$\left| t_{ij} \right| \gg \left| t_{kj} \right| \text{ for } k \neq j, \text{ and in the bandwidth of } t_{jj}$$  \hspace{1cm} (23)

Note that the above expression is scale invariant and is typically fulfilled once the MIMO system has been ordered according to appropriate methods like the Relative Gain Analysis, etc. Then the diagonal elements $t_{jj}$ will be much larger than the non-diagonal ones $t_{kj}$,

$$\left| t_{ij} \right| \gg \left| t_{kj} \right| \text{ for } k \neq j, \text{ and in the bandwidth of } t_{jj}$$  \hspace{1cm} (24)

Now, two simplifications are applied to facilitate the quantification of the coupling effects $c_{1ij}$, $c_{2ij}$, $c_{3ij}$.

**Simplification S1**: Using the Hypothesis H1, Eqs. (9), (16) and (22), which describe the coupling elements in the tracking problem, disturbance rejection at plant input and disturbance rejection at plant output respectively, are rewritten as shown Table I.

**Simplification S2**: The elements $t_{ij}$ are computed for each case from the equivalent system derived from Eqs. (6), (13) and (19). The results are shown in Table I.
Table I. Simplifications to quantify the coupling effects

<table>
<thead>
<tr>
<th>Simplification</th>
<th>Reference tracking</th>
<th>External disturbances at plant input</th>
<th>External disturbances at plant output</th>
</tr>
</thead>
<tbody>
<tr>
<td>S1</td>
<td>$c_{ij} = g_{ij} - t_{ij} \left( p_{ij}^* + g_{ij} \right)$ ; $i \neq j$</td>
<td>$c_{2ij} = t_{ij} \left( p_{ij}^* + g_{ij} \right)$ ; $i \neq j$</td>
<td>$c_{3ij} = p_{ij}^* - t_{ij} \left( p_{ij}^* + g_{ij} \right)$ ; $i \neq j$</td>
</tr>
<tr>
<td>S2</td>
<td>$t_{ij} = \frac{g_{jj} p_{jj}^{-1}}{1 + g_{jj} p_{jj}^{-1}}$ (28)</td>
<td>$t_{ij} = \frac{p_{jj}^{-1}}{1 + g_{jj} p_{jj}^{-1}}$ (29)</td>
<td>$t_{ij} = \frac{1}{1 + g_{jj} p_{jj}^{-1}}$ (30)</td>
</tr>
</tbody>
</table>

Due to Simplifications S1 and S2, the coupling effects $c_{1ij}$, $c_{2ij}$, $c_{3ij}$ can be computed as,

Tracking

$$c_{1ij} = g_{ij} - \frac{g_{jj} \left( p_{ij}^* + g_{ij} \right)}{p_{ij}^* + g_{jj}} \quad ; \quad i \neq j \quad (31)$$

Disturbance rejection at plant input

$$c_{2ij} = \frac{\left( p_{ij}^* + g_{ij} \right)}{p_{ij}^* + g_{jj}} \quad ; \quad i \neq j \quad (32)$$

Disturbance rejection at plant output

$$c_{3ij} = p_{ij}^* - \frac{p_{jj}^* \left( p_{ij}^* + g_{ij} \right)}{p_{ij}^* + g_{jj}} \quad ; \quad i \neq j \quad (33)$$

2.3 The Optimum Non-diagonal Controller

Consider non-diagonal controllers to reduce the coupling effect and diagonal controllers that help to achieve the loop performance specifications. The optimum non-diagonal controllers for the three cases (tracking and disturbance rejection at plant input and output) can be obtained making the loop interaction of Eqs. (31), (32) and (33) equal to zero.

Note that both elements, $p_{ij}^*$ and $p_{ij}^*$, of these equations are uncertain elements of $P^*$. Every uncertain plant $p_{ij}^*$ can be any plant represented by the family,

$$\{p_{ij}^*\} = p_{ij}^{*N} \left( 1 + \Delta_{ij} \right) , \quad 0 \leq |\Delta_{ij}| \leq \Delta p_{ij}^* , \quad \text{for } i, j = 1, \ldots, n \quad (34)$$

where $p_{ij}^{*N}$ is the nominal plant, and $\Delta p_{ij}^*$ the maximum of the non-parametric uncertainty radii $|\Delta_{ij}|$.

The nominal plants $p_{ij}^{*N}$ and $p_{ij}^{*N}$ that will be chosen for the optimum non-diagonal controller will
follow the next rules:

a) If the uncertain parameters of the plants show a uniform Probability Distribution (Fig. 5a) –which is typical in the QFT methodology–, then the elements $p_{ij}^{*}$ and $p_{jj}^{*}$ for the optimum non-diagonal controller will be the nominal plants $p_{ij}^{*N}$ and $p_{jj}^{*N}$, which minimise the maximum of the non-parametric uncertainty radii $\Delta p_{ij}^{*}$ and $\Delta p_{jj}^{*}$ that comprise the plant templates (Fig. 5b).

b) If the uncertain parameters of the plants show a non-uniform Probability Distribution (Fig. 5c), then the elements $p_{ij}^{*}$ and $p_{jj}^{*}$ for the optimum non-diagonal controller will be the nominal plants $p_{ij}^{*N}$ and $p_{jj}^{*N}$, whose set of parameters maximize the area of the Probability Distribution in the regions $[a_{ij} - \epsilon, a_{ij} + \epsilon]$ and $[a_{jj} - \epsilon, a_{jj} + \epsilon]$ ($\forall$ parameter $a_{ij}, b_{ij}, ..., a_{jj}, b_{jj} ...$) respectively.

Now, making Eqs. (31), (32) and (33) equal to zero and using Eq. (34), the optimum non-diagonal controller for each case is obtained.
2.3.1 Tracking

\[ g_{ij}^{\text{opt}} = F_{pd} \left( g_{jj}^{*N} \frac{p_{ij}^{*N}}{p_{jj}^{*N}} \right) , \text{ for } i \neq j \]  

(35)

2.3.2 Disturbance rejection at plant input

\[ g_{ij}^{\text{opt}} = F_{pd} \left( -p_{ij}^{*N} \right) , \text{ for } i \neq j \]  

(36)

2.3.3 Disturbance rejection at plant output

\[ g_{ij}^{\text{opt}} = F_{pd} \left( g_{jj}^{*N} \frac{p_{ij}^{*N}}{p_{jj}^{*N}} \right) , \text{ for } i \neq j \]  

(37)

where the function \( F_{pd}(A) \) means in every case a casual and stable proper function made from the dominant poles and zeros of the expression \( A \).

2.4 The Coupling Effects

The minimum achievable coupling effects -Eqs. (38), (40), (42)- can be computed substituting the optimum controller of Eqs. (35), (36) and (37) in the coupling expressions of Eqs. (31), (32) and (33) respectively, and taking into account the uncertainty radii of Eq. (34). Analogously, the maximum coupling effect without any non-diagonal controller -pure diagonal controller cases- can be computed substituting \( g_{ij}=0 \) in the Eqs. (31), (32) and (33) respectively -Eqs. (39), (41), (43)-. That is to say,

2.4.1 Tracking

\[ k_{1ij}^{\text{opt}} = |g_{ij} (\Delta_{ij} - \Delta_y) g_{ij}| \]  

(38)

\[ k_{1ij}^{\text{opt}} = |g_{ij} (1 + \Delta_y) g_{ij}| \]  

(39)

2.4.2 Disturbance rejection at plant input

\[ k_{2ij}^{\text{opt}} = |g_{ij} \Delta_y| \]  

(40)

\[ k_{2ij}^{\text{opt}} = |g_{ij} (1 + \Delta_y)| \]  

(41)

2.4.3 Disturbance rejection at plant output

\[ k_{3ij}^{\text{opt}} = |g_{ij} (\Delta_y - \Delta_y) g_{ij}| \]  

(42)

\[ k_{3ij}^{\text{opt}} = |g_{ij} (1 + \Delta_y) g_{ij}| \]  

(43)
where,

\[
\psi_{ij} = \alpha \frac{p_{ij}^\alpha}{(1 + \Delta_{ij}) p_{ij}^\alpha + g_{ij}}
\]

(44)

and the uncertainty is: \(0 \leq \Delta_{ij} \leq \Delta p_{ij}^\ast\), \(0 \leq \Delta_{ij} \leq \Delta p_{ij}^\ast\), for \(i, j = 1, ..., n\).

The coupling effects, calculated in the pure diagonal controller cases, result in three expressions (39), (41) and (43) that still present a non-zero value when the nominal-actual plant mismatching due to the uncertainty disappears: \(\Delta_{ij} = 0\) and \(\Delta_{ij} = 0\). However, the coupling effects obtained with the optimum non-diagonal controllers -Eqs. (38), (40) and (42)- tends to zero when that mismatching disappears.

2.5 Design Methodology

The proposed controller design methodology is a sequential procedure closing loops with four steps [2-7]:

**Step A: Controller structure, input-output pairing and loop ordering.** First, the methodology identifies the controller structure (minimum required elements of the controller matrix) and the input-output pairings by using the frequency-dependent Relative Gain Array -RGA- [10-11]. Then, the matrix \(P(s)\) is reorganized so that \(\{p_{11}(s)\}^{-1}\) has the smallest phase margin frequency, \(\{p_{22}(s)\}^{-1}\) the next smallest phase margin frequency, and so on to guarantee the existence of a solution [1].

After that, the sequential design technique composed of \(n\) stages, as many as loops, performs the following two steps \(B\) and \(C\) for every column of the matrix compensator \(G(s)\) from \(k = 1\) to \(n\) (Fig. 6).

\[
G = \begin{bmatrix}
ge_{11} & 0 & ... & 0 & 0 \\
g_{21} & g_{11} & 0 & ... & 0 \\
... & ... & ... & ... & ... \\
g_{n1} & g_{n2} & ... & 0 & 0 \\
\end{bmatrix} \Rightarrow \begin{bmatrix}
g_{11} & g_{12} & ... & 0 & 0 \\
g_{21} & g_{22} & ... & 0 & 0 \\
... & ... & ... & ... & ... \\
g_{n1} & g_{n2} & ... & 0 & 0 \\
\end{bmatrix} \Rightarrow \begin{bmatrix}
g_{11} & g_{12} & ... & g_{1k} & 0 \\
g_{21} & g_{22} & ... & g_{2k} & 0 \\
... & ... & ... & ... & ... \\
g_{n1} & g_{n2} & ... & g_{nk} & 0 \\
\end{bmatrix}
\]

Fig. 6 Steps for controllers design

**Step B: Design of the diagonal compensator \(g_{kk}(s)\).** The diagonal element \(g_{kk}(s)\) is calculated through standard QFT loop-shaping [1] for the inverse of the equivalent plant \(\{p_{kk}^\ast (s)\}^{-1}\) in order to achieve robust stability and robust performance specifications [13-14]. The equivalent plant satisfies the recursive relationship (45) [13], which is an extension for the non-diagonal case of the recursive expression proposed by Horowitz [12] as the Improved design technique, also called Second method by Houpis et al. [1].

\[
\left[p_{ij}^\ast(s)\right]_k = p_{ij}^\ast(s) = \left[\left[p_{i(k-1)}^\ast(s)\right]_{k-1} + \left[p_{i(k-1)}^\ast(s)\right]_{k-1} + [g_{i(k-1)}(s)]_{k-1}\right] + \left[p_{j(k-1)}^\ast(s)\right]_{k-1} + [g_{j(k-1)}(s)]_{k-1} \\
\]

(45)

\[i, j \geq k; \left[p_{ij}^\ast(s)\right]_{k-1} = P^\ast(s)\]
If the control system requires tracking specifications as \( a_{ii}(\omega) \leq |h_{ii}^{yt}(j\omega)| \leq b_{ii}(\omega) \) then, because \( t_{ii}^{yt} = t_{ri} + t_{cii} \) - Eq. (5) -, the tracking bounds \( b_{ii} \) and \( a_{ii} \) will have to be corrected with the coupling specification \( \tau_{ci} \), so that:

\[
\begin{align*}
    b_{ii}^c &= b_{ii} - \tau_{ci} \\
    t_{ci} &= w_{ci} \quad \tau_{ci} \leq \tau_{ci} \\
    a_{ii}^c(\omega) &\leq |t_{ri}^{yt}(j\omega)| \leq b_{ii}^c(\omega)
\end{align*}
\] (46) (47) (48)

These are the same corrections proposed originally by Horowitz (see also [1]). However, with the proposed non-diagonal method these corrections will be less demanding. The coupling expression \( \tau_{ci} = w_{ci} \) is now minor than in the previous diagonal methods - compare Eqs. (38) and (39) -. The off-diagonal elements \( g_{ik} (i \neq j) \) of the matrix controller will attenuate or cancel that cross coupling. Then the diagonal elements \( g_{kk} \) of the non-diagonal method will need less bandwidth than the diagonal elements of the previous diagonal methods.

**Step C: Design of the (n-1) non-diagonal elements \( g_{ik}(s) \) (i \neq k, i = 1,2,...,n).** The \( g_{ik}(s) \) (i \neq k) elements of the \( k \)-th compensator column are designed to minimize the non-diagonal elements of the cross-coupling matrices according to different purposes: reference tracking (31), (35); disturbance rejection at plant input (32), (36); and disturbance rejection at plant output (33), (37). The resulting compensators \( g_{ik}(s) \) have to be causal and stable, and include the dominant dynamics.

The off-diagonal controller elements can be allocated not only to reduce the coupling effects of the MIMO system, but also to reach complementary objectives, such as to remove RHP (right-half plane) transmission zeros introduced during the controller design [5], improve system integrity [13] and stability margins, reduce controller efforts, etc.

**Step D: Design of the prefilter.** The design of the prefilter \( F(s) \) does not present any additional difficulty because the final transfer function that relates \( R(s) \) to \( Y(s) \) shows less loop interaction thanks to the fully populated compensator design. Therefore, the prefilter \( F(s) \) can generally be a diagonal matrix.

### 2.6 Stability Conditions

Closed-loop stability of a MIMO system with a non-diagonal controller designed by using a sequential procedure is guaranteed by the following sufficient conditions [14]:

(c.1) each \( L_i(s) = g_{ii}(s) [p_{ii}^{-1}(s)]^{-1}, i=1, ..., n \), satisfies the Nyquist encirclement condition,

(c.2) no RHP pole-zero cancellations occur between \( g_{ii}(s) \) and \([p_{ii}^{-1}(s)]^{-1}, i=1, ..., n,\)

(c.3) no Smith-McMillan pole-zero cancellations occur between \( P(s) \) and \( G(s) \), and

(c.4) no Smith-McMillan pole-zero cancellations occur in \( \left| P^*(s) + G(s) \right| \).

### 2.7 Remarks

It is important to note that the calculation of the equivalent plant \([p_{kk}^{-1}(s)]_k^{-1}, (45)\), usually introduces some exact pole-zero cancellations. That operation could be precisely performed by using symbolic mathematical tools [1]. However, fictitious poles and zeros may be introduced when using numerical
calculus due to the typical rounding errors of the computer. Additionally, it is needed to determine the inverse of the plant matrix, which can also be numerically non-reliable.

In this paper, these problems are overcome through a new frequency response computation method. That is, for each frequency of interest $\omega$ and for every set of parameters within the region of uncertainty, each element $p_{ij}(j\omega)$ of the plant transfer function matrix is translated into a complex matrix $P_{freq,ij}$ that represents the frequency response of every plant element within the uncertainty. Thus, this complex matrix has as many rows as different cases generated due to the uncertainty and as many columns as frequencies (49). All the abovementioned calculations are then performed on the basis of this set of complex matrices by using element-by-element matrix operations. As a result, potential impediments related to practical computation are avoided.

$$a_{ij} = \text{Re}(a_{ij}) + j\text{Imag}(a_{ij})$$

At the same time, arbitrarily picking the wrong order of the loops to be designed can result in the non-existence of a solution. This may occur if the solution process is based on satisfying an upper limit of the phase margin frequency $\omega_{\phi}$ for each loop. Hence, Loop $i$ having the smallest phase margin frequency will have to be chosen as the first loop to be designed. The loop that has the next smallest phase margin frequency will be next, and so on [1].

Although very remote, theoretically there exists the possibility of introducing RHP transmission zeros due to the compensator design. This undesirable situation can not be detected until the multivariable system design is completed. To avoid it the proposed methodology (Steps A, B and C) is inserted in a procedure introduced by Garcia-Sanz and Eguinoa [5]. Once the matrix compensator $G(s)$ is designed, the transmission zeros of $P(s)$ are determined using the Smith-McMillan form and over the set of possible plants $\mathcal{P}$ due to uncertainty. If there exist new RHP transmission zeros apart from those initially present in $P(s)$, they can be removed by using the non-diagonal elements placed in the last column of the matrix $G(s)$.

### 3.0 MIMO QFT CONTROL OF SPACECRAFT WITH FLEXIBLE APPENDAGES [7]

This section summarizes the design of a robust non-diagonal MIMO QFT controller to govern the position and attitude of a Darwin-type spacecraft with large flexible appendages. The satellite is one of the flyers of a multiple spacecraft constellation for a future ESA mission. It presents a 6x6 high order MIMO model with large uncertainty and loop interactions introduced by the flexible modes of the low-stiffness appendages. The scientific objectives of the satellite require very demanding control specifications for position and attitude accuracy, high disturbance rejection, loop-coupling attenuation and low order controller. This section demonstrates the feasibility of sequential non-diagonal MIMO QFT strategies.
controlling the Darwin spacecraft and compares the results with a previous H-infinity design.

3.1 Description

The Darwin mission consists of three to six telescopes arranged in a symmetric configuration flying in formation around a master satellite or central hub (Fig. 7). Darwin will employ nulling interferometry to detect and analyze through appropriate spectroscopy techniques the atmosphere of remote planets close to a bright star. The infrared light collected by the free flying telescopes will be recombined inside the hub-satellite in such a way that the light from the central star suffers destructive interference and is cancelled out, allowing this way the much fainter planet easier to stand out. The interferometry requires very accurate and stable positioning of the spacecraft in the constellation, which puts high demands on the attitude and position control system. Darwin will be placed further away, at a distance of 1.5 million kilometers from Earth, in the opposite direction from the Sun (Earth-Sun Lagrangian Point L2 –Fig.8).

![Fig. 7 Darwin spacecraft (Artist's view. ESA courtesy)](image)

![Fig. 8 Earth-Sun Lagrangian Points and Darwin spacecraft location](image)

Each telescope flyer is cylindrically shaped (2 m diameter, 2 m height) and weighs 500 kg. In order to protect the instrument from the sunlight, it is equipped with a sunshield modeled with 6 large flexible
beams (4 m long and 7 kg) attached to the rigid structure (Fig. 9; beam end-point coordinates in brackets). The main mechanical characteristics of the Darwin-type Flyer are summarized in Table II.

For every beam (Fig. 9), two different frequencies for the first modes along Y and Z beam axes are considered. Their frequency can vary from 0.05 Hz to 0.5 Hz, with a nominal value of 0.1 Hz, and their damping can vary from 0.1% to 1%, with a nominal value of 0.5%. As regards spacecraft mass and inertia, the corresponding uncertainty around their nominal value is of 5%.

Based on the previous description and using a mechanical modeling formulation for multiple flexible appendages of a rigid body spacecraft, the open-loop transfer function matrix representation of the Darwin-type Flyer is given in (50) and Fig. 10:

\[
\begin{bmatrix}
\dot{x}(s) \\
\dot{y}(s) \\
\dot{z}(s) \\
\dot{\varphi}(s) \\
\dot{\theta}(s) \\
\dot{\psi}(s)
\end{bmatrix} = P(s) \cdot U(s) = \\
\begin{bmatrix}
p_{11}(s) & p_{12}(s) & p_{13}(s) & p_{14}(s) & p_{15}(s) & p_{16}(s) \\
p_{21}(s) & p_{22}(s) & p_{23}(s) & p_{24}(s) & p_{25}(s) & p_{26}(s) \\
p_{31}(s) & p_{32}(s) & p_{33}(s) & p_{34}(s) & p_{35}(s) & p_{36}(s) \\
p_{41}(s) & p_{42}(s) & p_{43}(s) & p_{44}(s) & p_{45}(s) & p_{46}(s) \\
p_{51}(s) & p_{52}(s) & p_{53}(s) & p_{54}(s) & p_{55}(s) & p_{56}(s) \\
p_{61}(s) & p_{62}(s) & p_{63}(s) & p_{64}(s) & p_{65}(s) & p_{66}(s)
\end{bmatrix} \\
\begin{bmatrix}
F_x(s) \\
F_y(s) \\
F_z(s) \\
T_\varphi(s) \\
T_\theta(s) \\
T_\psi(s)
\end{bmatrix}
\] (50)

where \( x, y, z \) are the position coordinates; \( \varphi, \theta, \psi \) are the corresponding attitude angles; \( F_x, F_y, F_z \) are the force inputs; \( T_\varphi, T_\theta, T_\psi \) are the torque inputs; and where every \( p_{ij}(s), i, j = 1, \ldots, 6 \), is a 50th order Laplace transfer function with uncertainty.

Fig. 9  Darwin type 6 DOF satellite model

The Bode diagram of the plant (Fig. 10) shows the dynamics of the 36 matrix elements. Each of them and the MIMO system (matrix) are minimum phase. The flexible modes introduced by the appendages (second-order dipoles) affect all the elements around the frequencies \( \omega = [0.19, 10] \text{ rad/sec} \). The diagonal elements \( p_{ii}(s), i = 1, \ldots, 6 \), and the elements \( p_{15}(s), p_{33}(s), p_{24}(s) \) and \( p_{42}(s) \) are mainly double integrators plus the flexible modes.

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Table II. Mechanical characteristics of the Darwin-type Flyer model

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Satellite body mass</td>
<td>500 kg</td>
</tr>
<tr>
<td>Cylinder dimensions</td>
<td>2 m diameter, 2 m height</td>
</tr>
<tr>
<td>Inertia tensor of satellite in control frame at satellite Centre of Mass (without reflector)</td>
<td>$\mathbf{I}<em>{\text{Darwin}</em>\text{body,Cont}_\text{CoM}} = \begin{bmatrix} 250 &amp; 0 &amp; 0 \ 0 &amp; 250 &amp; 0 \ 0 &amp; 0 &amp; 250 \end{bmatrix}$ kg \cdot m$^2$</td>
</tr>
<tr>
<td>Inertia tensor of satellite in control frame at satellite Centre of Mass (with reflector)</td>
<td>$\mathbf{I}<em>{\text{Darwin}</em>\text{body,Cont}_\text{CoM}} = \begin{bmatrix} 509 &amp; 0 &amp; 0 \ 0 &amp; 509 &amp; 0 \ 0 &amp; 0 &amp; 684 \end{bmatrix}$ kg \cdot m$^2$</td>
</tr>
<tr>
<td>Position of Centre of Mass in control frame at satellite Centre of Mass</td>
<td>[0, 0, 0] m</td>
</tr>
<tr>
<td>Sunshield mass</td>
<td>7 kg * 6 beams = 42 kg</td>
</tr>
<tr>
<td>Beam length</td>
<td>4 m</td>
</tr>
</tbody>
</table>

Fig. 10 Darwin-type flyer dynamics

The block diagram of the control system is shown in Fig. 11. The sensor module represents both the OPD (Optical Pathlength Differences) Fringe Tracker sensor and the FPM (Fine Pointing Metrology) sensor, which measure the satellite position and attitude, respectively. The actuators, FEEP (Field Emission Electric Propulsion) thrusters, are a type of electrostatic propulsion that provides very small and precise actuation (Table III).
The external disturbances acting on the satellite (gravity gradient and solar pressure), although very small, are also modeled as forces and torques along the 3 axes. The gravity gradient is modeled as a constant bias and the solar pressure is represented as a white noise perturbation (Table III).

The original dynamics benchmark simulator, provided by ESA and implemented under Matlab/Simulink, integrates all those elements constituting the whole satellite control system: sensors, actuators, dynamics, disturbances, etc. (Fig. 11). For each performance evaluation campaign, 300 random dynamics within the uncertainty (Monte-Carlo analysis) are generated to evaluate the performance of the controllers.
Table IV. Darwin-type Flyer requirements

<table>
<thead>
<tr>
<th>Objective</th>
<th>Numerical Requirement</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Astronomical Requirements</strong></td>
<td></td>
</tr>
<tr>
<td>Position accuracy</td>
<td>Maximum absolute value: 1 µm for all axes</td>
</tr>
<tr>
<td></td>
<td>Standard deviation: 0.33 µm for all axes</td>
</tr>
<tr>
<td>Pointing accuracy</td>
<td>Maximum absolute value: 25.5 mas for all axes (3 σ)</td>
</tr>
<tr>
<td></td>
<td>Standard deviation: 8.5 mas for all axes (1 σ)</td>
</tr>
<tr>
<td><strong>Engineering Requirements</strong></td>
<td></td>
</tr>
<tr>
<td>Bandwidth</td>
<td>~ 0.01 Hz for all axes</td>
</tr>
<tr>
<td>Saturation limits</td>
<td>Maximum force: 150 µN</td>
</tr>
<tr>
<td></td>
<td>Maximum torque: 150 µNm</td>
</tr>
<tr>
<td>Rejection of high frequency noises (from measurement and actuation)</td>
<td>High roll-off after the bandwidth</td>
</tr>
<tr>
<td><strong>Control Requirements</strong></td>
<td></td>
</tr>
<tr>
<td>Stability margins</td>
<td>$\max_{\omega}</td>
</tr>
<tr>
<td></td>
<td>$\max_{\omega}</td>
</tr>
<tr>
<td>Loop interaction</td>
<td>Minimum</td>
</tr>
<tr>
<td>Rejection of flexible modes</td>
<td>Maximum</td>
</tr>
<tr>
<td>Controller complexity and order</td>
<td>Minimum</td>
</tr>
</tbody>
</table>

3.2 Control objectives

The main objective of the spacecraft is to fulfill some astronomical requirements that demand to keep the flying telescope pointing at both the observed space target and the central hub-satellite. This set of specifications leads to some additional engineering requirements (bandwidth, saturation limits, noise rejection, etc.) and also needs some complementary control requirements (stability, low loop interaction, low controller complexity and order, etc.) –Table IV-.

3.3 Non-diagonal MIMO QFT Controller Design

The sequential non-diagonal MIMO QFT methodology previously described in Section 2 [2-7] is applied here to control the position and attitude of the Darwin-type Flyer.

3.3.1 Relative Gain Array Interaction Analysis –Step A-

The Relative Gain Array (RGA) of a non-singular square matrix $P$ is a scale-invariant measure of interactions. Its original definition introduced by Bristol [11] was only proposed for steady state ($\omega = 0$ rad/sec). However, the RGA can also be computed frequency-by-frequency (51) and used to assess the interaction at frequencies other than zero [10]. In most cases, the value of RGA at frequencies close to crossover is the most important one.

$$RGA(j\omega) = \left[ \lambda_{ij}(j\omega) \right] = P(j\omega) \otimes \left( P^{-1}(j\omega) \right)^T$$  \hspace{1cm} (51)

where $\otimes$ denotes element-by-element multiplication (Schur product). Further information on how to interpret the RGA results and select pairings can be found at [10, 11].

The 6x6 (position and attitude) dynamic model of the Darwin-type spacecraft with large flimsy appendages has been analyzed by using the RGA method as a function of frequency and for the whole set of parameter uncertainty. Although the matrix obtained by means of (51) is a complex one, only the
absolute values are presented. By examining the corresponding matrices at each frequency, it is observed
that the steady state results extend through low frequency up to 0.19 rad/sec. As a representative example
within this range, (52) shows the results for the most coupled plant within the uncertainty at \( \omega = 6.28\times10^{-4} \)
rad/sec. According to it, the pairing should be done through the main diagonal of the matrix, which
contains positive RGA elements, and the elements \( g_{15}(s) \), \( g_{24}(s) \), \( g_{42}(s) \), and \( g_{51}(s) \) should also be considered
relevant.

\[
RGA_{(\omega=6.28\times10^{-4}\text{rad/sec})} = \begin{bmatrix}
1.0064 & 0 & 0 & 0 & 0.0064 & 0 \\
0 & 1.0064 & 0 & 0.0064 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0.0064 & 0 & 1.0064 & 0 & 0 \\
0.0064 & 0 & 0 & 0 & 1.0064 & 0 \\
0 & 0 & 0 & 0 & 0 & 1
\end{bmatrix} \quad (52)
\]

If the analysis is performed at high frequency, it produces the same concluding results in the entire
spectrum starting at 3.51 rad/sec.

So far, the retained compensator elements would be those of the RGA matrix marked in bold in (52).
Nevertheless, as aforementioned, the RGA elements increase and more interactions are founded in the
interval of frequencies where the flexible modes of the satellite mostly affect (i.e. \( \omega = [0.19-3.51] \) rad/sec),
as can be seen in (53) and (54) for the most coupled plants at \( \omega = 0.8 \) rad/sec and \( 1 \) rad/sec, respectively.

\[
RGA_{(\omega=0.8\text{rad/sec})} = \begin{bmatrix}
1.1674 & 0.0001 & 0.0000 & 0.0012 & 0.3720 & 0.0001 \\
0.0001 & 1.0180 & 0.0000 & 0.0305 & 0.0013 & 0.0004 \\
0.0000 & 0.0000 & 4.8592 & 0.3468 & 4.1456 & 0.0000 \\
0.0012 & 0.0035 & 0.3468 & 3.2030 & 2.3865 & 0.0006 \\
0.3720 & 0.0013 & 4.1456 & 2.3865 & 7.2470 & 0.0008 \\
0.0001 & 0.0004 & 0.0000 & 0.0006 & 0.0008 & 1.0009
\end{bmatrix} \quad (53)
\]

\[
RGA_{(\omega=1\text{rad/sec})} = \begin{bmatrix}
0.9899 & 0.0002 & 0.0000 & 0.0001 & 0.0003 & 0.0102 \\
0.0002 & 0.9082 & 0.0000 & 0.0050 & 0.0009 & 0.0963 \\
0.0000 & 0.0000 & 3.1307 & 3.3373 & 0.1110 & 0.0000 \\
0.0001 & 0.0050 & 2.3373 & 12.5674 & 9.4613 & 0.0690 \\
0.0003 & 0.0009 & 0.1110 & 9.4613 & 10.2042 & 0.0302 \\
0.0102 & 0.0963 & 0.0000 & 0.0690 & 0.0302 & 0.7970
\end{bmatrix} \quad (54)
\]

### 3.3.2 Controller Structure

In accordance with the above RGA results and taking into account the requirement of minimum controller
complexity and order (Table IV), a first compensator structure consisting of six diagonal elements and
four off-diagonal elements is chosen as the most suitable one (55).

\[
G(s) = \begin{bmatrix}
g_{11}(s) & 0 & 0 & 0 & g_{15}(s) & 0 \\
0 & g_{22}(s) & 0 & g_{24}(s) & 0 & 0 \\
0 & 0 & g_{33}(s) & 0 & 0 & 0 \\
0 & g_{42}(s) & 0 & g_{44}(s) & 0 & 0 \\
g_{51}(s) & 0 & 0 & 0 & g_{55}(s) & 0 \\
0 & 0 & 0 & 0 & 0 & g_{66}(s)
\end{bmatrix} \quad (55)
\]
From this, four independent compensator design problems have been adopted, two SISO and two 2x2 MIMO problems: \[ g_{33}(s) \text{ and } g_{66}(s); \ g_{11}(s) \ g_{15}(s) \ g_{51}(s) \ g_{55}(s) \text{ and } g_{22}(s) \ g_{24}(s) \ g_{42}(s) \ g_{44}(s) \], respectively. The SISO problems will be considered from the classical SISO QFT point of view, while the two 2x2 MIMO subsystems will be studied through the non-diagonal MIMO QFT methodology. The coupling detected in the range of frequencies of the flexible modes will be considered in the course of the design procedure through more demanding specifications with respect to disturbance rejection. Provided the performance results were not satisfactory, then the compensator structure should be filled up with additional off-diagonal compensators consistent with (53) and (54): \( g_{34}, g_{35}, g_{43}, g_{45}, g_{53} \text{ and } g_{54} \). 

### 3.3.3 Robust Closed-Loop Specifications

The applied non-diagonal and diagonal MIMO QFT techniques design each loop individually, including the multivariable characteristic by means of their corresponding equivalent plant. So, the robust closed-loop specifications are defined in terms of SISO expressions for both SISO and MIMO subsystems.

Since these methodologies are frequency domain techniques, there is obviously a need for translating time domain requirements (Table IV) into the frequency domain. The original specifications in Table IV impose maximum and standard deviation values on the position and attitude time error signals, as well as actuator forces and torques. Their translation into the frequency domain is based on control-ratio models [9], and takes into account the expected external disturbances on the Darwin-type flyer, the spacecraft flexible modes and the coupling among loops. As a result, four Type of specifications are defined to calculate the QFT bounds: Type 1: Robust stability; Type 2: Robust sensitivity; Type 3: Robust disturbance rejection at plant input; and Type 4: Robust control effort attenuation.

The notation used for the signals in the following description of specifications refers to the scheme of the generic MIMO subsystem presented in Fig. 12. The compensators have been designed within the set of frequencies of interest \( \omega = [6.28 \times 10^{-4}, 62.8] \) rad/sec.

![Fig. 12 Structure of a 2 Degree of Freedom MIMO System](image)

**Type 1: Robust Stability specification**

This specification, shown in (56), is stated to guarantee a robust stable control. All the required values, displayed loop by loop in (57) and (58), imply at least 1.54 (3.75 dB) gain margin and at least 49.25° phase margin. The specification corresponds not only to the closed-loop transfer function \( y_i(s)/r_i(s) \), but also to transfer functions \( y_i(s)/n_i(s) \) and \( u_i(s)/n_i(s) \). Hence this condition additionally imposes the requirements on sensor noise attenuation, disturbance rejection at plant input and flexible modes.

\[
\left| \frac{p_{u}^{-1} e_{u}}{1 + p_{u}^{-1} e_{u}^{-1}} \right| \leq \delta_1(\omega) \tag{56}
\]

where \( [p_{u}^{-1} e_{u}(s)]^{-1} \) is the inverse of the equivalent plant (45), which corresponds to \( p_{i}(s) \) in SISO designs.
Loops 1, 2 and 3: \[ \delta_1(\omega) = 1.85 \quad ; \quad \forall \omega \] \hfill (57)

Loops 4, 5 and 6: \[ \delta_1(\omega) = \left| \frac{0.1687}{s^2 + 0.4s + 0.0912} \right| \quad ; \quad \forall \omega \] \hfill (58)

**Type 2: Sensitivity reduction**

The main objective of this specification, (59) and (60), is sensor noise attenuation and reduction of the effect of the parameter uncertainty on the closed-loop transfer function. It corresponds to \( e_i(s)/n_i(s) \) and \( [d_{ii}(s)/t_{ii}(s)] / [p_{ii}(s)/p_{ii}(s)] \) transfer functions.

\[ \left| \frac{1}{1 + \left[ \frac{1}{p_{ii}(s)} \right]^{-1} g_u(s)} \right| \leq \delta_2(\omega) \] \hfill (59)

All loops: \[ \delta_2(\omega) = 2 \quad ; \quad \forall \omega \] \hfill (60)

**Type 3: Rejection of disturbances at plant input**

Solar pressure perturbation and gravity gradient are considered to affect at plant input in the form of both force and torque. The purpose of this specification (61), which corresponds to \( e_i(s)/n_i(s) \) and \( y_i(s)/v_i(s) \) transfer functions, is to attenuate the effect of plant input disturbances on the control error and the output signal. Thus, a high gain is required in the low frequency band, (62) to (64). Besides, since \( v_i(s) \) also represents the flexible modes, special attention is paid to their frequency range mainly to accomplish the attitude requirements.

\[ \left| \frac{1}{1 + \left[ \frac{1}{p_{ii}(s)} \right]^{-1} g_u(s)} \right| \leq \delta_3(\omega) \] \hfill (61)

Loops 1 and 2: \[ \delta_3(\omega) = \left| \frac{0.21553 \left( s + 0.385 \right)}{(s + 0.307)(s + 6.18) \left( s^2 + 0.4s + 0.0912 \right)} \right| \quad ; \quad \forall \omega \] \hfill (62)

Loop 3: \[ \delta_3(\omega) = \left| \frac{0.313 \left( s - 0.01705 \right)(s^2 + 0.009974s + 5.104 \times 10^{-5})}{(s - 0.01813)(s^2 + 0.02554s + 0.0004754)} \right| \quad ; \quad \forall \omega \] \hfill (63)

Loops 4, 5 and 6: \[ \delta_3(\omega) = \left| \frac{(s + 0.2)(s + 0.186)(s + 0.2044)(s + 0.003892)(s^2 + 0.06014s + 0.02736)}{(s + 0.007333)(s + 0.445)(s^2 + 0.07904s + 0.00326)(s^2 + 0.2352s + 0.0981)} \right| \quad ; \quad \forall \omega \] \hfill (64)

**Type 4: Control signal**

Because of saturation limits, control signal movements should be kept reasonably small despite disturbances coming from actuators and sensors. This specification, (65), corresponds to \( u_i(s)/n_i(s) \) transfer function and is depicted in (66)-(68).

\[ \left| \frac{g_u(s)}{1 + \left[ \frac{1}{p_{ii}(s)} \right]^{-1} g_u(s)} \right| \leq \delta_4(\omega) \] \hfill (65)
Loops 1 and 2: \[
\delta_4(\omega) = \frac{557.1 (s + 5)}{(s^2 + 3.23s + 6.5)}; \quad \forall \omega
\]

Loop 3: \[
\delta_4(\omega) = \frac{106.9210 (s + 0.55)(s^2 + 0.04s + 0.13)}{(s + 1.4)^2 (s^2 + 0.1227s + 0.097)}; \quad \forall \omega
\]

Loops 4, 5 and 6: \[
\delta_4(\omega) = \frac{4.026 (s^2 - 0.1854s + 0.203)(s^2 + 0.04s + 0.504)}{(s^2 + 0.305s + 0.056)(s^2 + 0.115s + 0.095)}; \quad \forall \omega
\]

Reducing coupling effects as much as possible

The coupling effects from other axes can be considered as part of the disturbances acting at the input of the equivalent SISO plant. The way of designing the non-diagonal elements of the matrix compensator deals with the aim of minimizing the off-diagonal elements of the coupling matrix (32).

3.3.4 SISO Design Problems: \(g_{33}(s), g_{66}(s)\)

Compensators \(g_{33}(s)\) and \(g_{66}(s)\) are independently designed by using classical SISO QFT [1] to satisfy the robust stability and robust performance specifications stated in Section 3.3.3 for every plant within the set of uncertain plants. The corresponding QFT bounds and the nominal case of the designed open-loop transfer functions \(L_i(s) = p_i(s) g_i(s), i = 3, 6,\) are plotted on the Nichols Chart for some of the most relevant frequencies in Fig. 13(a) and 13(b) for loops 3 and 6 respectively. Both designs satisfy not only their respective bounds but also the Nyquist encirclement condition, and no RHP pole-zero cancellations occur between \(g_{33}(s)\) and \(p_{33}(s)\), nor between \(g_{66}(s)\) and \(p_{66}(s)\). The Bode plot of each compensator can be found in Section 3.6, where \(g_{33}(s)\) [Fig. 18(a)] and \(g_{66}(s)\) [Fig. 18(b)] are represented in solid line in comparison with the H-infinity design (dashed line) introduced in Section 3.5. The QFT compensator expressions are presented in Section 3.5.

![Loop-Shaping L_{33}](image-a)  
![Loop-Shaping L_{66}](image-b)

Fig. 13 Loop-shaping (a) \(L_{33}(s) = p_{33}(s) g_{33}(s)\), (b) \(L_{66}(s) = p_{66}(s) g_{66}(s)\)
3.3.4.1 First MIMO Problem: \(g_{11}(s), g_{51}(s), g_{55}(s), g_{15}(s)\) Design

The compensator for this 2x2 MIMO problem has been designed by applying the non-diagonal MIMO QFT methodology developed by Garcia-Sanz et al. [1-7] and outlined in Section 2. In this particular case, the plant to be controlled is composed of the following elements coming from the original 6x6 Darwin-type spacecraft model \(P^{15}(s) = [p_{11}(s) \ p_{15}(s) ; p_{51}(s) \ p_{55}(s)]\), whose inverse matrix is \(P^{15*}(s) = [P^{15}(s)]^{-1} = [p_{11}^*(s) \ p_{15}^*(s) ; p_{51}^*(s) \ p_{55}^*(s)]\).

**Step A: Arrangement of the system**

First, the methodology adopts the structure and the pairing of inputs and outputs given by the RGA technique in (55) and arranges the plant inverse matrix \(P^{15*}(s)\) so that the inverse of the first diagonal element in this matrix has the smallest phase margin frequency [1]. In some cases, arbitrarily picking the wrong order of the loops could lead to the non-existence of a solution. In the present problem, the bandwidth of the loops is quite similar. Then, any order can be selected to design the non-diagonal MIMO QFT compensators.

**Step B1: Design of the diagonal compensator \(g_{11}(s)\)**

The diagonal compensator \(g_{11}(s)\) is designed through standard QFT loop-shaping [1] for the inverse of the equivalent plant \(\left[p_{11}^*(s)\right]_1 = p_{11}(s)\) to fulfill the robust stability and robust performance specifications determined in Section 3.3.3 for every plant within the set of uncertain plants. Fig. 14(a) shows the nominal case of the designed open-loop transfer function \(L_{11}(s) = \left[p_{11}^*(s)\right]_1^{-1} g_{11}(s)\) in bold solid line, which satisfies the QFT bounds, also represented in the figure. Additionally, the design fulfills the first two sufficient stability conditions (c.1) and (c.2) (Section 2.6). That is, \(L_{11}(s) = \left[p_{11}^*(s)\right]_1^{-1} g_{11}(s)\) satisfies the Nyquist encirclement condition and no RHP pole-zero cancellations occur between \(g_{11}(s)\) and \(\left[p_{11}^*(s)\right]_1^{-1}\). The Bode plot for the obtained compensator \(g_{11}(s)\) is presented in Fig. 19(a) (solid line) together with the design of the H-infinity approach.

**Step C1: Design of the non-diagonal compensator \(g_{51}(s)\)**

The non-diagonal compensator \(g_{51}(s)\) is designed to minimize the (5,1) element of the coupling matrix in the case of disturbance rejection at plant input (32), which gives the following expression:
where $N$ denotes the plant that minimizes the maximum of the non-parametric uncertainty radii comprising the plant templates on the Nichols Chart. Due to the uncertainty, the expression $[-p_{51}^*(s)]$ determines a region in the magnitude and phase plots, where the compensator $g_{51}(s)$ is shaped following the mean value at every frequency $\omega \in [0, 0.1]$ rad/sec [see Fig. 15 with $g_{51}(s)$ interpolating the plot]. The compensator Bode plot is compared in Fig. 19(c) with that of the $(5,1)$ element of the H-infinity compensator introduced in Section 3.5.

Step B2: Design of the diagonal compensator $g_{55}(s)$

As in step B1, the diagonal compensator $g_{55}(s)$ is designed to control the inverse of the equivalent plant, $[p_{55}^{*e}(s)]_2^{-1}$, which takes the compensator previously designed into account (45).

$$
[p_{55}^{*e}(s)]_2^{-1} = \frac{[p_{51}^{*e}(s)]_1 + [g_{51}(s)]_1}{[p_{11}^{*e}(s)]_1 + [g_{11}(s)]_1}
$$

(70)

On the basis of the robust specifications defined in Section 3.3.3 for $[p_{55}^{*e}(s)]_2^{-1}$, and also taking into account the plant uncertainty, the QFT bounds are computed. Then, the compensator is designed by classical loop-shaping on the Nichols Chart, as is shown in Fig. 14(b). Not only does the design fulfil the bounds but also the first two stability conditions of (c.1) and (c.2) from Section 2.6. In other words, $L_{55}(s) = [p_{55}^{*e}(s)]_2^{-1} g_{55}(s)$ satisfies the Nyquist encirclement condition and no RHP pole-zero cancellations occur between $g_{55}(s)$ and $[p_{55}^{*e}(s)]_2^{-1}$. The Bode plot of $g_{55}(s)$ is presented in Fig. 19(d).

Step C2: Design of the non-diagonal compensator $g_{15}(s)$

Due to the requirement of minimum controller complexity and order (Table IV), the non-diagonal compensator $g_{15}(s)$ has been set to zero. Anyway, the equivalent expression to the one used in (69), $g_{15}^{opt}(s) = -p_{15}^{*N}(s)$, could be applied to cancel interaction in both directions in the MIMO subsystem.

At this point, once the whole controller of the MIMO subsystem has been determined, the last two stability conditions mentioned in Section 2.6, (c.3) and (c.4), are checked. The system is stable. Finally,
the non-existence of RHP transmission zeros of \( P(s) G(s) \) is checked by using the Smith-McMillan form over the set of possible plants \( \mathcal{P} \) due to uncertainty [5]. The non-diagonal MIMO QFT compensator expressions are presented in Section 3.6.

### 3.3.4.2 Second MIMO Problem: \( g_{22}(s), g_{42}(s), g_{44}(s), g_{24}(s) \) Design

The second MIMO problem consists of the following elements: \( P^{24}(s) = [p_{22}(s) \ p_{24}(s) \ p_{42}(s) \ p_{44}(s)] \). From the 2x2 plant inverse matrix \( P^{24*}(s) = [P^{24}(s)]^{-1} = [p_{22}^{-1}(s) \ p_{24}^{-1}(s) \ p_{42}^{-1}(s) \ p_{44}^{-1}(s)] \) and taking into account the robust stability and robust performance specifications (Section 3.3.3), the non-diagonal MIMO QFT methodology is equivalently performed by following the steps detailed in Section 2.

The loop-shaping for the diagonal compensator elements \( g_{22}(s) \) and \( g_{44}(s) \) are shown in Fig. 16(a) and 16(b), respectively. The Bode plots for the four compensators are shown in Fig. 20(a), (b), (c) and (d) for \( g_{22}(s), g_{24}(s), g_{42}(s), g_{44}(s) \), respectively. The 2x2 MIMO subsystem is found to be stable according to the sufficient stability conditions (Section 2.6). Finally, it is also checked that no additional RHP zeros have been introduced by the compensator [5]. The non-diagonal MIMO QFT compensator expressions are presented in Section 3.6.

![Fig. 16 Loop-shaping](a) \( L_{22}(s) = [p_{22}^{-1}(s)]^{-1} g_{22}(s) \) (b) \( L_{44}(s) = [p_{44}^{-1}(s)]^{-1} g_{44}(s) \)

### 3.4 Diagonal MIMO QFT Controller Design

For the sake of comparison, the sequential diagonal MIMO QFT methodology developed by Horowitz [12] is also applied to control the position and attitude of the Darwin-type Flyer. Based on the same robust closed-loop specifications defined in Section 3.3.3, this technique uses a sequential procedure similar to the one detailed in Section 2.5 (Step B), where the recursive expression of the equivalent plant is a simplified case of (45), with \( g_{ij}(s) = 0 \) \( (i \neq j) \).

For the Darwin-type Flyer, the loop-shaping step of the diagonal method requires the same diagonal compensators \( g_{ii}(s) \) as the non-diagonal one. This happens because, in this case, in the middle and high frequency range the off-diagonal elements \( g_{ij}(s) \) \( (i \neq j) \) of the non-diagonal controller have less relative influence than the corresponding \( p_{ij}(s) \) elements in the equivalent plant (45). Differences between both MIMO QFT controllers arise in the low frequency range, as can be observed in Fig. 17. The \( C_{2}(5,1) \) element of the coupling matrix for disturbances at plant input (32) is plotted for a representative plant case.
and for the three controllers: non-diagonal and diagonal MIMO QFT and H-infinity designs. For the frequency range \( \omega \in [0, 0.1] \) rad/sec it is shown that \( C_{2}(5,1)_{\text{non-diag QFT}} < C_{2}(5,1)_{\text{diag QFT}} < C_{2}(5,1)_{\text{H-infinity}} \), which explains why the non-diagonal MIMO QFT improves the diagonal MIMO QFT and the H-infinity controller results under low frequency external disturbances (see Section 3.6.2).

**Fig. 17** Element (5,1) of the coupling matrix \( C_{2} \): non-diagonal MIMO QFT in solid line, diagonal MIMO QFT in dotted line, H-infinity in dashed line

### 3.5 Controllers

The notation adopted for transfer function expressions denotes the steady state gain as a constant without parenthesis; simple poles and zeros as \((\omega)\), which corresponds to \((s/\omega + 1)\) denominator and numerator, respectively; poles and zeros at the origin as \((0)\); conjugate poles and zeros as \([\xi; \omega_{n}]\), with \(\left((s/\omega_{n})^{2} + (2\xi/\omega_{n})s + 1\right)\) denominator and numerator, each; \(n\)-multiplicity of poles and zeros as an exponent \((\_)^{n}\).

The non-diagonal MIMO QFT compensator consists of the following eight elements:

\[
g_{11}(s) = g_{22}(s) = \frac{31.5 (0.6194) (0.2138) (0.1663) (0.1649)}{(0.666) (0.4982) (0.07526) (0.676; 1.4791)}; \quad g_{33}(s) = \frac{125 (0.13) (0.057) [0.07019; 0.3565]}{(1.48) (0.7875) (0.2) (0.004) (0.00246) (0.18; 0.3141)}; \quad g_{44}(s) = \frac{2.242 (0.03412) (0.08644; 0.7114)}{(1.48) (0.7875) (0.2) (0.004) (0.00246) (0.18; 0.3141)}
\]

The diagonal MIMO QFT compensator consists of the same diagonal elements \(g_{ii}(s)\) as the non-diagonal compensator abovementioned, and \(g_{ij}(s) = 0, i \neq j\).

The main elements of the 1-DOF H-infinity compensator are shown in Figs. 18-20. Their dc gains stay within the range [-15 dB, 26 dB]. The remaining 26 elements present a very low gain, going from -260 dB to -330 dB.
Fig. 18 Bode Diagram Compensators: non-diagonal and diagonal MIMO QFT in solid line, H-infinity in dashed line. (a) $g_{33}(s)$, (b) $g_{66}(s)$

Fig. 19 Bode Diagram Compensators: non-diagonal MIMO QFT in solid line [also diagonal MIMO QFT for $g_{11}(s)$ and $g_{55}(s)$], H-infinity in dashed line. (a) $g_{11}(s)$, (b) $g_{15}(s)$, (c) $g_{51}(s)$, (d) $g_{55}(s)$
3.6 Comparative evaluation

This section shows a comparative analysis of the sequential non-diagonal MIMO QFT controller, designed above for the 6x6 Darwin-type Flyer, with both sequential diagonal MIMO QFT and H-infinity controllers. First, comparative Bode plots of the compensators are shown. Then, time performance results are presented (astronomical requirements), followed by open-loop bandwidth, and forces and torques comparison (engineering requirements). Finally, the stability objectives and the order of each compensator are analyzed (control requirements).

3.6.1 Compensators Bode Plots

The Bode plots are presented for the compensators of the non-diagonal MIMO QFT (solid line) in comparison with those of the H-infinity (dashed line). Note that, in this case, the diagonal MIMO QFT method yields the same diagonal compensators as the non-diagonal MIMO QFT technique. Fig. 18 presents the results for the two SISO subsystems $g_{33}(s)$ and $g_{66}(s)$, (a) and (b) respectively. Fig. 19 plots
the compensators of the 2x2 MIMO subsystem composed of $g_{11}(s)$, $g_{15}(s)$, $g_{51}(s)$ and $g_{55}(s)$ elements. The $g_{22}(s)$, $g_{24}(s)$, $g_{42}(s)$ and $g_{44}(s)$ compensator elements that conform the other 2x2 MIMO subsystem are shown in Fig. 20. Note that $g_{15}(s)$ and $g_{24}(s)$ have been set to zero in the non-diagonal MIMO QFT design. Additionally, according to the RGA results in (55) the remaining elements of the controller matrix $G(s)$ designed with this technique equal zero. By contrast, those 26 elements present a non-zero, although very small, magnitude response when they are designed with the H-infinity technique. Finally, the off-diagonal elements of the diagonal MIMO QFT are obviously zero.

Table V. Time simulation performance with the three controllers

<table>
<thead>
<tr>
<th>Specification</th>
<th>Requirement</th>
<th>Benchmark</th>
<th>Non-diagonal MIMO QFT Controller</th>
<th>Diagonal MIMO QFT Controller</th>
<th>H-infinity Controller</th>
</tr>
</thead>
<tbody>
<tr>
<td>1 Maximum Position Error X ($\mu$m)</td>
<td>&lt; 1 $\mu$m</td>
<td>B1</td>
<td>0.0131</td>
<td>0.0131</td>
<td>0.0293</td>
</tr>
<tr>
<td></td>
<td></td>
<td>B2</td>
<td>0.0816</td>
<td>0.0816</td>
<td>0.511</td>
</tr>
<tr>
<td>2 Maximum Position Error Y ($\mu$m)</td>
<td>&lt; 1 $\mu$m</td>
<td>B1</td>
<td>0.0120</td>
<td>0.0120</td>
<td>0.0299</td>
</tr>
<tr>
<td></td>
<td></td>
<td>B2</td>
<td>0.0120</td>
<td>0.0120</td>
<td>0.0299</td>
</tr>
<tr>
<td>3 Maximum Position Error Z ($\mu$m)</td>
<td>&lt; 1 $\mu$m</td>
<td>B1</td>
<td>0.0288</td>
<td>0.0288</td>
<td>0.0292</td>
</tr>
<tr>
<td></td>
<td></td>
<td>B2</td>
<td>0.0288</td>
<td>0.0288</td>
<td>0.0292</td>
</tr>
<tr>
<td>4 Maximum Attitude Error X (mas)</td>
<td>&lt; 25.5 mas</td>
<td>B1</td>
<td>25.27</td>
<td>25.31</td>
<td>25.95</td>
</tr>
<tr>
<td></td>
<td></td>
<td>B2</td>
<td>25.27</td>
<td>25.31</td>
<td>25.95</td>
</tr>
<tr>
<td>5 Maximum Attitude Error Y (mas)</td>
<td>&lt; 25.5 mas</td>
<td>B1</td>
<td>22.91</td>
<td>22.99</td>
<td>23.21</td>
</tr>
<tr>
<td></td>
<td></td>
<td>B2</td>
<td>22.55</td>
<td>23.75</td>
<td>28.91</td>
</tr>
<tr>
<td>6 Maximum Attitude Error Z (mas)</td>
<td>&lt; 25.5 mas</td>
<td>B1</td>
<td>21.15</td>
<td>21.15</td>
<td>22.84</td>
</tr>
<tr>
<td></td>
<td></td>
<td>B2</td>
<td>21.15</td>
<td>21.15</td>
<td>22.84</td>
</tr>
<tr>
<td>7 Std. Deviation of Position Error X ($\mu$m)</td>
<td>&lt; 0.33 $\mu$m</td>
<td>B1</td>
<td>0.00275</td>
<td>0.00276</td>
<td>0.00686</td>
</tr>
<tr>
<td></td>
<td></td>
<td>B2</td>
<td>0.0511</td>
<td>0.0511</td>
<td>0.341</td>
</tr>
<tr>
<td>8 Std. Deviation of Position Error Y ($\mu$m)</td>
<td>&lt; 0.33 $\mu$m</td>
<td>B1</td>
<td>0.00265</td>
<td>0.00266</td>
<td>0.00722</td>
</tr>
<tr>
<td></td>
<td></td>
<td>B2</td>
<td>0.00265</td>
<td>0.00266</td>
<td>0.00722</td>
</tr>
<tr>
<td>9 Std. Deviation of Position Error Z ($\mu$m)</td>
<td>&lt; 0.33 $\mu$m</td>
<td>B1</td>
<td>0.00668</td>
<td>0.00668</td>
<td>0.00691</td>
</tr>
<tr>
<td></td>
<td></td>
<td>B2</td>
<td>0.00668</td>
<td>0.00668</td>
<td>0.00691</td>
</tr>
<tr>
<td>10 Std. Deviation of Attitude Error X (mas)</td>
<td>&lt; 8.5 mas</td>
<td>B1</td>
<td>5.57</td>
<td>5.57</td>
<td>5.68</td>
</tr>
<tr>
<td></td>
<td></td>
<td>B2</td>
<td>5.57</td>
<td>5.57</td>
<td>5.68</td>
</tr>
<tr>
<td>11 Std. Deviation of Attitude Error Y (mas)</td>
<td>&lt; 8.5 mas</td>
<td>B1</td>
<td>5.76</td>
<td>5.76</td>
<td>6.01</td>
</tr>
<tr>
<td></td>
<td></td>
<td>B2</td>
<td>5.80</td>
<td>5.85</td>
<td>8.23</td>
</tr>
<tr>
<td>12 Std. Deviation of Attitude Error Z (mas)</td>
<td>&lt; 8.5 mas</td>
<td>B1</td>
<td>4.83</td>
<td>4.83</td>
<td>5.00</td>
</tr>
<tr>
<td></td>
<td></td>
<td>B2</td>
<td>4.83</td>
<td>4.83</td>
<td>5.00</td>
</tr>
<tr>
<td>13 Maximum Actuator Force Command X (N)</td>
<td>&lt; 1.5e-4 N</td>
<td>B1</td>
<td>1.94e-6</td>
<td>1.94e-6</td>
<td>7.42e-7</td>
</tr>
<tr>
<td></td>
<td></td>
<td>B2</td>
<td>3.94e-6</td>
<td>3.94e-6</td>
<td>3.31e-6</td>
</tr>
<tr>
<td>14 Maximum Actuator Force Command Y (N)</td>
<td>&lt; 1.5e-4 N</td>
<td>B1</td>
<td>1.86e-6</td>
<td>1.86e-6</td>
<td>6.68e-7</td>
</tr>
<tr>
<td></td>
<td></td>
<td>B2</td>
<td>1.86e-6</td>
<td>1.86e-6</td>
<td>6.68e-7</td>
</tr>
<tr>
<td>15 Maximum Actuator Force Command Z (N)</td>
<td>&lt; 1.5e-4 N</td>
<td>B1</td>
<td>5.94e-7</td>
<td>5.94e-7</td>
<td>5.61e-7</td>
</tr>
<tr>
<td></td>
<td></td>
<td>B2</td>
<td>5.94e-7</td>
<td>5.94e-7</td>
<td>5.61e-7</td>
</tr>
<tr>
<td>16 Maximum Actuator Torque Command X (Nm)</td>
<td>&lt; 1.5e-4 N m</td>
<td>B1</td>
<td>8.68e-7</td>
<td>8.71e-7</td>
<td>1.03e-6</td>
</tr>
<tr>
<td></td>
<td></td>
<td>B2</td>
<td>8.68e-7</td>
<td>8.71e-7</td>
<td>1.03e-6</td>
</tr>
<tr>
<td>17 Maximum Actuator Torque Command Y (Nm)</td>
<td>&lt; 1.5e-4 N m</td>
<td>B1</td>
<td>1.05e-6</td>
<td>1.05e-6</td>
<td>1.15e-6</td>
</tr>
<tr>
<td></td>
<td></td>
<td>B2</td>
<td>1.06e-6</td>
<td>1.06e-6</td>
<td>1.16e-6</td>
</tr>
<tr>
<td>18 Maximum Actuator Torque Command Z (Nm)</td>
<td>&lt; 1.5e-4 N m</td>
<td>B1</td>
<td>1.08e-6</td>
<td>1.08e-6</td>
<td>1.27e-6</td>
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<tr>
<td></td>
<td></td>
<td>B2</td>
<td>1.08e-6</td>
<td>1.08e-6</td>
<td>1.27e-6</td>
</tr>
</tbody>
</table>
3.6.2 Astronomical Requirements

Time simulations are performed for 300 random dynamics within the uncertainty range (MonteCarlo analysis) in the original ESA benchmark simulator (B1) described in Section 3.1 and in a complementary benchmark (B2), both developed under Matlab/Simulink. The latter just adds to B1 a low frequency disturbance force at plant input along the X-axis (magnitude: 2 \( \mu \)N, frequency: \( \omega = 0.05 \) rad/sec) in order to consider disturbance rejection and coupling attenuation at low frequencies. In each simulation, the criteria appearing in Table IV are computed over the entire simulation time (i.e. 5000 sec).

In order to characterize the minimum performance obtained, the worst results reached by every controller are presented in Table V. In other words, for each controller, the greatest value over the 300 uncertain cases is shown for the maximum and the standard deviation of position and attitude errors, as well as for maximum actuator commands, in all axes. Then, it is possible to verify whether the worst performance still respects the requirement. The bold number in every row of Table V represents the best result (best control strategy) for every particular specification.

**Position errors (1,2,3,7,8,9 –Table V)**

By inspecting Table V, it is found that the performance obtained in time simulations is very good concerning position accuracy, since the requirements are easily fulfilled (an improvement of two orders of magnitude with respect to the specification is achieved in benchmark B1, and at least one order of magnitude in benchmark B2) for maximum and standard deviation values. The non-diagonal MIMO QFT design either equals or slightly improves the diagonal MIMO QFT. Both QFT controllers improve the H-infinity results for the two benchmarks.

**Attitude errors (4,5,6,10,11,12 –Table V)**

The specification for the highest attitude error is harder to meet mainly because of the effect of the flexible modes. Some of the maximum attitude values of the H-infinity even exceed the 25.5 mas required: see benchmark B1 (4 –Table V) and benchmark B2 (4,5 –Table V). Again, the MIMO QFT methodologies improve the results of the H-infinity controller in the six attitude error cases (4,5,6,10,11,12 –Table V).

Once more, the non-diagonal MIMO QFT either equals or improves the diagonal QFT controller results. The greatest differences between both controllers can be observed at the Attitude Error along the Y-axis (5,11 –Table V), especially for benchmark B2. There, the non-diagonal design decreases the standard deviation attitude error by 0.85 % (11 –Table V) and the maximum attitude error by 5.05 % (5 –Table V) with respect to the values reached by the diagonal compensator. These improvements could turn out to be relevant to the astronomical mission. Their achievement is due to the fact that the off-diagonal compensators have been designed to minimize the coupling at low frequencies, which are principally stressed in the second benchmark.

3.6.3 Engineering Requirements

**Saturation limits. Actuator commands (13,14,15,16,17,18 –Table V)**

As can be seen in Table V, actuation is very small and far below the saturation limits. The results for the three controllers remain at similar values (13,14,15,16,17,18 –Table V).

**Open-loop Bandwidth Comparison**

The open-loop cross-over frequency results of the six SISO loop subsystems are shown in Table VI. These measures correspond to the smallest frequencies in Hz where the transfer functions of the open-loop of each SISO subsystem \( p_{ii}(s) g_{ii}(s) \) [without the coupling elements \( p_{ij}(s), i \neq j \)] are equal to 0 dB. The
minimum performance for each of the three designs has been established as the minimum bandwidth value over the 300 random satellite dynamics. Obviously, the bandwidth results for the two MIMO QFT designs coincide since their diagonal compensators are the same. A value of 0.01 Hz is considered a good compromise choice for bandwidth. Since the frequencies of the first flexible modes are within the range [0.05, 0.5] Hz, the open-loop cross-over frequencies for attitude and position are tuned to be as high as possible while simultaneously preventing the flexible modes from disturbing the system output performance.

### Table VI. Frequency performance with the three controllers

<table>
<thead>
<tr>
<th>Requirement</th>
<th>Non-diagonal MIMO QFT Controller</th>
<th>Diagonal MIMO QFT Controller</th>
<th>H-infinity Controller</th>
</tr>
</thead>
<tbody>
<tr>
<td>Position Bandwidth X (Hz)</td>
<td>0.01 Hz</td>
<td>0.0464</td>
<td>0.0464</td>
</tr>
<tr>
<td>Position Bandwidth Y (Hz)</td>
<td>0.01 Hz</td>
<td>0.0472</td>
<td>0.0472</td>
</tr>
<tr>
<td>Position Bandwidth Z (Hz)</td>
<td>0.01 Hz</td>
<td>0.0212</td>
<td>0.0212</td>
</tr>
<tr>
<td>Attitude Bandwidth X (Hz)</td>
<td>0.01 Hz</td>
<td>0.00949</td>
<td>0.00949</td>
</tr>
<tr>
<td>Attitude Bandwidth Y (Hz)</td>
<td>0.01 Hz</td>
<td>0.0102</td>
<td>0.0102</td>
</tr>
<tr>
<td>Attitude Bandwidth Z (Hz)</td>
<td>0.01 Hz</td>
<td>0.00883</td>
<td>0.00883</td>
</tr>
</tbody>
</table>

| max\(|P(j\omega)| (dB)\) | <6 dB                             | 12.73 (Max)                  | 11.51 (Max)           | 4.70 (Max)            |
|--------------------------|-----------------------------------|------------------------------|-----------------------|-----------------------|
| max\(|S(j\omega)| (dB)\) | <6 dB                             | 13.11 (Max)                  | 12.05 (Max)           | 6.24 (Max)            |

For the position transfer functions, the three controllers exceed the 0.01 Hz recommendation (two and even four times depending on the controller and the axis). However, the flexible modes mostly affect the attitude transfer functions and do not impose such strong constraints on the position transfer functions. Consequently, it is possible to go over 0.01 Hz for the position loops, as is proved by the satisfying time domain results in Table V.

For the attitude transfer functions, the open-loop cross-over frequencies are around 0.01 Hz for the H-infinity and for both the non-diagonal and diagonal MIMO QFT designs.

### 3.6.4 Control Requirements

#### Stability Objectives

Stability and performance specifications are essentially described as mathematical expressions ready to be used during the design process of the controller. These expressions usually differ from one control methodology to another provided they are based on distinct approaches, which is the case of H-infinity and QFT-based methodologies. In this paper the stability specifications have been defined in two different ways:

a) Stability conditions of Section 2.6 for MIMO QFT (Nyquist criterion for sequential methods).
b) Margins on max\(|P(j\omega)|\) and max\(|S(j\omega)|\) for H-infinity (classical criterion for MIMO systems).

The non-diagonal and the diagonal MIMO QFT controllers fulfill the stability conditions for sequential procedures defined in Section 2.6. The H-infinity compensator fulfills the margins of \(T(s)\) and \(S(s)\) defined in Table VI. With respect to this classical interpretation of robust stability, the QFT approaches respect them in most of the cases (mean), but not in several cases (max). This is due to the fact that these interpretations of the stability margins (which are indeed a margin of a margin) are not integrated as a design specification in the core of QFT techniques.
Since stability and performance specifications are only interpretations of the functional requirements (astronomical and engineering requirements –Table IV-), the designer should be aware of which tradeoffs need to be made. Essentially, the interpretation of reality in terms of a particular theory can never replace the real world itself. In the absence of the real system implementation or a suitable prototype to be used instead, the designer must manage time domain simulations in order to verify the control system behavior [1]. That interpretation was done and successfully validated in the previous sections of the paper.

Additionally, the classical margins on $\max_{\omega} \left| T(j\omega) \right|$ and $\max_{\omega} \left| S(j\omega) \right|$ are stability MIMO margins, but they do not include phase information. This fact makes them sufficient, but not necessary conditions and could yield very conservative controllers in some situations. Although the three methods are robust stable according to their own requirement and to time domain simulations, future research work to re-interpret both types of robust stability conditions and margins constitute one of the next research objectives.

**Controller Complexity and Order**

The number of operations that have to be performed per sampling period may place restrictions on the compensator design. The implementation of a controller based on the state space representation differs from that based on transfer functions. The former appears to have a common denominator for every element of the compensator when it is transformed into transfer function description and the latter does not actually need it. Indeed, the expression of each control signal in a transfer function matrix depends on its corresponding row of the compensator matrix. But even there, common denominators are not needed. The control signal $u_i(s)$ is computed as the sum of signals generated by every compensator in the $i$-th row.

In order to make a realistic comparison of the computational cost of the different controllers (non-diagonal MIMO QFT, H-infinity and diagonal MIMO QFT), the number of sums and multiplications computed in each sample at the final implementation are analyzed. Following the same discretization process, the values in Table VII are obtained. The compensator matrix of the H-infinity design expressed in transfer function description presents 36 elements having 42nd order. The diagonal MIMO QFT design consists of six diagonal compensators going from 5th to 14th order. The non-diagonal MIMO QFT design consists of eight compensators going from 3rd to 14th order.

<table>
<thead>
<tr>
<th>Controller</th>
<th>Number of Multiplications</th>
<th>Number of Sums</th>
</tr>
</thead>
<tbody>
<tr>
<td>Non-diagonal MIMO QFT</td>
<td>130</td>
<td>124</td>
</tr>
<tr>
<td>H-infinity</td>
<td>2994</td>
<td>2988</td>
</tr>
<tr>
<td>Diagonal MIMO QFT</td>
<td>116</td>
<td>110</td>
</tr>
</tbody>
</table>

**Table VII.** Computational cost per sampling for the three controllers
4.0 MIMO QFT CONTROL OF FORMATION FLYING IN LOW EARTH ORBIT [8]

This section applies the reformulation of the non-diagonal MIMO QFT methodology (which also considers matrix specifications) (see [8] for details) to control a spacecraft flying in formation with respect to a central body in a circular low Earth orbit with uncertainty. The example presents some extra difficulties such as RHP zeros, poles on the imaginary axis, required RHP transmission zeros in the compensator, uncertainty, etc.

4.1 Motivation

In the last few decades, there has existed a growing interest in control of multiple spacecraft flying in formation, which was initiated by Wiltshire and Clohessy with their early works about rendezvous. That interest is shown by the increasing published literature on the subject; see for example the recent special issues of the *IET Control Theory and Applications* journal, edited by Garcia-Sanz, and of the *International Journal of Robust and Nonlinear Control*, edited by Rasmussen and Schumacher, etc.

With the development of small satellites, formation flying has become a cornerstone for present and future ESA and NASA missions. Among its numerous applications are variable-baseline interferometry, large-scale distributed sensors and sparse antenna arrays. It will also allow the replacement of expensive multiple-payload platforms with co-observatories consisting of several low cost spacecraft.

4.2 Model

The non-linear dynamic equation of motion of a spacecraft flying in formation, relative to the centre of mass of the formation (c.m.f), following a closed Keplerian reference orbit, is shown in (72),

\[
m \dot{\rho} + 2m [\omega] \dot{\rho} + m (([\omega]^{2} + [\omega]) \dot{\rho} + m (\dot{b} + [\omega] b) = Q + D - U_q \tag{72}
\]

where the reference orbit defines an orbit reference frame (x,y,z) – (Figure 21), which serves as the primary frame to analyse the dynamic of the spacecraft formation.

The unit vector \(x(o_1)\) points anti-nadir, \(z(o_3)\) points in the direction of the orbit normal, and \(y(o_2)\) completes the right-handed triad. The inertial frame of reference \(n_1, n_2, n_3\) – (Figure 21)-, is attached to the centre of the Earth. The unit vector \(n_1\) points toward the vernal equinox, \(n_3\) points toward the geographic North Pole, and \(n_2\) completes the right-handed triad. \(m\) is the mass of the spacecraft; \(Q = [Q_x, Q_y, Q_z]^T\) are the actuator forces (thrusters); \(D = [D_x, D_y, D_z]^T\) are the disturbance forces acting on the satellite due to luni-solar attraction, solar radiation pressure, Earth’s oblatness, tidal effects, Earth’s geomagnetic field, etc.; \(U_q = \frac{\partial}{\partial q} \left( -\frac{\mu m}{R} \right)\) is the gradient of the Earth potential field; \([\omega] = [0 -\omega_0 0; \omega_0 0 0; 0 0 0]^T\); \(b = [\dot{R}_0, R_0 \omega_0, 0]^T\), \(R = [(x + R_0)^2 + y^2 + z^2]^{1/2}\).
The system outputs to be controlled are the distances $\rho = [x, y, z]^T$; $\dot{\rho} = x \dot{O}_1 + y \dot{O}_2 + z \dot{O}_3$ of the spacecraft to the c.m.f (orbit reference frame). $R_0$ is the mean orbit radius, $\omega_0 = \sqrt{\mu / R_0^3}$ is the mean orbit rate and $\mu = 3.9860 \times 10^5$ Km$^3$/sec$^2$ denotes the gravitational parameter of the Earth. Both parameters ($R_0$ and $\omega_0$) present some uncertainty due to the disturbances. Alternatively, the reference orbit can also be described by the orbital elements $a$ (semi-major axis), $e$ (eccentricity), $i_1$ (inclination), $\Omega_1$ (longitude of the ascending node), $\omega_1$ (argument of perigee), and $T$ (time of perigee passage) – Figure 21-.

For circular orbits, the orbit radius $R_0$ can present a slow time-variation due to uncertainty, which means $\dot{R}_0 \approx 0$ and $\ddot{R}_0 \approx 0$. Applying these conditions in (72), the relative equations of motion of a spacecraft relative to the c.m.f in circular orbit are,

$$\ddot{x} - 2 \omega_0 \dot{x} - 3 \omega_0^2 x = \frac{1}{m} (Q_x + D_x)$$ (73)

$$\ddot{y} + 2 \omega_0 \dot{y} = \frac{1}{m} (Q_y + D_y)$$ (74)

$$\ddot{z} + \omega_0^2 z = \frac{1}{m} (Q_z + D_z)$$ (75)

which are the so-called Clohessy-Wiltshire-Hill (CWH) equations.
Applying the Laplace Transform on these equations, the transfer function matrix $P$ relating the actuator forces and the position of the spacecraft becomes,

$$
\begin{bmatrix}
x \\
y \\
z
\end{bmatrix}
= \frac{1}{m}
\begin{bmatrix}
x
\end{bmatrix}
\begin{bmatrix}
Q_x \\
Q_y \\
Q_z
\end{bmatrix}
= \frac{1}{m}
\begin{bmatrix}
\frac{1}{s^2 + \omega_0^2} & \frac{2 \alpha_0}{s (s^2 + \omega_0^2)} & 0 \\
-\frac{2 \alpha_0}{s (s^2 + \omega_0^2)} & \frac{s^2 - 3 \omega_0^2}{s^2 (s^2 + \omega_0^2)} & 0 \\
0 & 0 & \frac{1}{s^2 + \omega_0^2}
\end{bmatrix}
\begin{bmatrix}
Q_x \\
Q_y \\
Q_z
\end{bmatrix}
$$

(76)

which is a 2x2 MIMO system (axes $x$ and $y$) plus a SISO system (axis $z$).

Note that although the $p_{22}$ element of the system is non-minimum phase [see (76)], the MIMO plant does not present RHP transmission zeros. Its Smith-McMillan decomposition (77) yields a double integrator and a conjugate pole at $s = \pm j \omega_0$, but no transmission zeros.

$$
Smith-McMillan(P) = \begin{bmatrix}
1 \\
\frac{1}{s^2 (s^2 + \omega_0^2)} \\
0 \\
1
\end{bmatrix}
$$

(77)

### 4.3 Parameter values

Equation (76) is particularised for a spacecraft flying in formation in a circular geostationary Earth orbit. Since the sidereal day is 23 h. 56 min. 4 sec, then $\omega_0 = \frac{2 \pi}{86164 \text{ sec}} = 7.29212 \times 10^{-5} \text{ rad/sec}$, which corresponds to an orbit radius $R_0 = 42164 \text{ km}$. This radius presents some uncertainty because of external disturbances, so the same goes for the mean orbit rate $\omega_0$. The mass of the satellite presents uncertainty as well –see Table VIII-.

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Nominal value</th>
<th>Minimum value</th>
<th>Maximum value</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\omega_0$ (rad/sec)</td>
<td>$7.29212 \times 10^{-5}$</td>
<td>$7.2637 \times 10^{-5}$</td>
<td>$7.3208 \times 10^{-5}$</td>
</tr>
<tr>
<td>$R_0$ (km)</td>
<td>42164</td>
<td>42054</td>
<td>42274</td>
</tr>
<tr>
<td>$m$ (kg)</td>
<td>1650</td>
<td>1600</td>
<td>1650</td>
</tr>
</tbody>
</table>

### 4.4 Objectives and Specifications

The main objective of the controller is to keep the satellite stable, rejecting disturbances while controlling the distance $\rho$ between the spacecraft and the c.m.f., which corresponds to the nominal orbit trajectory.
Remember that the notation adopted for transfer function expressions denotes the steady state gain as a constant without parenthesis; simple poles and zeros as \((\omega)\), which corresponds to \((s/\omega + 1)\); poles and zeros at the origin as \((0)\); conjugate poles and zeros as \([\zeta; \omega_n]\), with \(((s/\omega_n)^2 + (2\zeta/\omega_n) s + 1)\); \(n\)-multiplicity of poles and zeros as an exponent \((n)\).

**Matrix specifications on \(S(s)\) and \(T(s)\)**

Classical robust MIMO specifications are defined in terms of the H-infinity norm of the matrices \(S(s)\) and \(T(s)\) (sensitivity and complementary sensitivity matrix, respectively). For this particular problem the following margins are selected:

\[
||T(j\omega)||_\infty < 2 \quad \text{and} \quad ||S(j\omega)||_\infty < 2
\]  

(78)

Then, (78) is translated into QFT loop-by-loop specifications according to the new methodology and for the sufficient condition. In order to put it into effect, the desired matrix performance specifications and their elements \(\tau = [\tau_{ij}]\) must be determined. The load is shared between the loops but also taking into account that (78) must be fulfilled through those specifications.

For stability, the upper limits for inequalities are defined:

\[
b_{11}' = b_{22}' = \frac{(0.00672)(0.14)(1.093)}{(0.011)(0.265)(0.35)(2)}
\]  

(79)

The corresponding coupling attenuation elements in the stability case are given by:

\[
\tau_{c11} = \tau_{c22} = \frac{10^{-6} (0.00672)(0.14)(1.093)}{(0.011)(0.265)(0.35)(2)}
\]  

(80)

\[
\tau_{c12} = 10^{-5}
\]  

(81)

\[
\tau_{c21} = -1.5 \times 10^{-5}
\]  

(82)

Using equations (79) and (80), the upper limits \(b_{11}\) and \(b_{22}\), respectively, can be straightforwardly obtained.

For disturbance rejection at plant output:
$$\tau_{so11} = \frac{0.00428 (0)(7.25 \times 10^{-5})^2 (8 \times 10^{-5})(2.08)}{(4 \times 10^{-4})^2 (0.0047)(0.4)(0.68)}$$

(83)

$$\tau_{so12} = \frac{0.903 (0)^2 (9 \times 10^{-5})^2 (0.0012)(0.0015)}{(3 \times 10^{-4})^2 (4 \times 10^{-4})(2 \times 10^{-4})(0.0013)(0.011)(0.45)^2}$$

(84)

$$\tau_{so21} = \frac{237.58(0)^2 (7 \times 10^{-5})(1.28 \times 10^{-4})(6.9 \times 10^{-4})(0.12)}{(2.2 \times 10^{-4})(2.11 \times 10^{-4})(2.3 \times 10^{-4})(2.4 \times 10^{-4})(2.16 \times 10^{-4})(0.026)(0.5)^2}$$

(85)

$$\tau_{so22} = \frac{2.376 \times 10^6 (0)^3 (8 \times 10^{-5})(2.31)}{(10^{-4})(2.6 \times 10^{-4})(0.01)(0.358)(0.84)}$$

(86)

These specification matrices yield the following singular value plot, where the maximum value within the frequency is 6 dB for both plots.

Fig. 22 Maximum singular value of the matrix specifications on T(s) and S(s)
Classical QFT robust stability and robust performance specifications

Additionally, to show the possibilities of the new method, three classical loop-by-loop QFT specifications are also defined, so that,

**Robust stability (Type 1)**

\[
\left| \frac{g_{ii}^\beta (s) \left[ (p_{ii}^x(s))^e \right]^{-1}}{1 + g_{ii}^\beta (s) \left[ (p_{ii}^x(s))^e \right]^{-1}} \right| \leq 1.1 \quad \forall \omega , \ i = 1,2,3
\]  

(87)

**Disturbance rejection at plant output (Type 2)**

To be consistent with the matrix specification on the \( S(s) \) matrix of the 2x2 MIMO subsystem, an additional specification on \( s_{33}(s) \) is considered.

\[
\left| \frac{1}{1 + g_{33}(s) p_{33}(s)} \right| \leq 2 \quad \forall \omega
\]  

(88)

**Disturbance rejection at plant input (Type 3)**

\[
\left| \frac{\left[ (p_{ii}^x(s))^e \right]^{-1}}{1 + g_{ii}^\beta (s) \left[ (p_{ii}^x(s))^e \right]^{-1}} \right| \leq T_d \quad \forall \omega < 1 \text{ rad/sec} , \ i = 1,2,3
\]  

(89)

where,

\[
T_d = \frac{20 (0)}{(0.008)(0.06)(0.6)}
\]  

(90)

The Bode magnitude plot of \( T_d \) is shown in Figure 23, and corresponds to the step time response of Figure 24.
4.5 Controller design

Open-loop properties

The RGA is computed as a function of frequency. The $\lambda_{11}$ element presents a value of -3 in low frequency, which changes to approximately 1 near $7 \times 10^{-5}$ rad/sec, maintaining this value within the high frequency range, see Fig. 24. According to the control properties of the RGA, this change of sign detects the presence of the RHP-zero in the $p_{22}$ element, see (76).
To decide which pairings must be chosen, it is taken into account the value of the RGA element around the expected cross-over frequency. In this case, this value is around 0.2 rad/sec, where the RGA adopts a value near 1. Thus, pairing is made along the diagonal elements.

**MIMO subsystem**

According to the desired specifications, the off-diagonal compensators design problem belongs to Case 1. Then, the methodology (see [8] for details) designs two controllers: \(G_\alpha\) and \(G_\beta\).

**Controller \(G_\alpha\)**

The controller \(G_\alpha\) is designed according to the new method. The analytical expression used as a starting point is,

\[
\hat{P}^{-1} \hat{P}_{\text{diag}} = \begin{bmatrix}
\frac{s^2 - 3\alpha^2}{s^2 + \alpha^2} & -2\alpha \frac{s^2 - 3\alpha^2}{s^2 + \alpha^2} & 0 \\
2\alpha s & \frac{s^2 - 3\alpha^2}{s^2 + \alpha^2} & 0 \\
0 & 0 & 1
\end{bmatrix}
\]  \(\text{(91)}\)

However, (91) is modified to avoid cancellations in the RHP and the imaginary axis, as well as RHP transmission zeros introduced by the \(G_\alpha\) compensator, but still decoupling the system and keeping the relative difference of the number of poles and zeros of the elements in (91). That is, its frequency response is approximated, the notion of all-pass filter elements is used, and some damping is introduced.

The final \(G_\alpha\) controller is shown in (92). The uncertain system is decoupled in low frequency, obtaining a positive value of \(\lambda_{11} = \lambda_{22}\) around 0.98. The same positive sign is kept for the high frequency RGA. So, the design of \(G_\alpha\) clearly improves the results obtained for the original plant \(P\) –

\[
G_\alpha = \begin{bmatrix}
\frac{3\sqrt{3}\alpha^2}{\xi \alpha} & -6\alpha \frac{3\alpha^2}{\xi \alpha} & 0 \\
2\alpha^{-1}(0) & \frac{3\sqrt{3}\alpha^2}{\xi \alpha} & 0 \\
0 & 0 & 1
\end{bmatrix}
\]  \(\text{(92)}\)

where \(\alpha = 7.2827 \times 10^{-5}\) rad/sec ; \(\zeta = 0.8\)
Controller $G_β$

The controller $G_β$, loops 1 and 2, is designed according to the method introduced for the extended plant $P^x = P G_α$.

Loop 1: Design of $g_{11}^β$

The compensator $g_{11}^β$, see (93), is calculated by using a sequential (loop-by-loop) standard QFT loop-shaping methodology for the inverse of the equivalent extended plant $[p_{11}^{x^c}])^{-1}$ which satisfies the recursive relationship, with $g_{ij} = 0$, $i \neq j$.

$$g_{11}^β = \frac{0.06 (0.018) (0.0077)}{(0) (1.25) (1.2)}$$  \hspace{1cm} (93)

![Necessary and sufficient bounds for matrix specification on S(s)](image)

Figure 25 Loop 1: Necessary and sufficient bounds for matrix specification on $S(s)$

Figure 25 shows a detail of the bounds when they are translated into necessary and sufficient conditions on the Nichols Chart for the matrix specification on $S(s)$. The designed loop transfer function $L_{11}(s) = [p_{11}^{x^c}(s)]^{-1} g_{11}^β(s)$ should be at every frequency above the solid lines or under the dashed lines. The gap between the necessary –red lines- and the sufficient –green lines- bounds is due to the uncertainty.

The loop-shaping of the open-loop transfer function $L_{11}(s)$ is shown in 26, where the nominal case is represented in solid bold line along with the sufficient QFT bounds, which are fulfilled for all the frequencies. The design also fulfils the first two sufficient stability conditions (i) and (ii). In other words, $L_{11}(s) = [p_{11}^{x^c}(s)]^{-1} g_{11}^β(s)$ satisfies the Nyquist encirclement condition and no RHP pole-zero cancellations occur between $g_{11}^β(s)$ and $[p_{11}^{x^c}(s)]^{-1}$.
Fig. 26 Loop-shaping \( L_{11}(s) = [p_{11}^{\times e}]_1^{-1} g_{11}^\beta \)

**Loop 2: Design of \( g_{22}^\beta \)**

The compensator \( g_{22}^\beta \), see (95), is calculated by using a sequential (loop-by-loop) standard QFT loop-shaping methodology for the inverse of the equivalent extended plant \( [p_{22}^{\times e}(s)]_2^{-1} \), in (94). It satisfies the recursive relationship with \( g_9 = 0, i \neq j \), and takes into account the compensator element \( g_{11}^\beta \) previously designed.

\[
[p_{22}^{\times e}(s)]_2 = [p_{22}^{\times e}(s)]_1 - \frac{[p_{21}^{\times e}(s)]_1 [p_{12}^{\times e}(s)]_1}{[p_{11}^{\times e}(s)]_1 + [g_{11}^\beta(s)]_1} \tag{94}
\]

\[
g_{22}^\beta = \frac{0.07 \times 0.022 \times 0.0069}{(0.022 \times 0.0069)(1.7)^2} \tag{95}
\]
The Nichols Chart in 27 represents the nominal open-loop transfer function $L_{22}(s) = [p_{22} x^e(s)]^2 - 1$ and the corresponding QFT bounds for this loop, which take into account the robust specifications and the uncertainty of the plant to be controlled, that is, $[p_{22} x^e(s)]^2 - 1$. Additionally, the design fulfills the stability conditions (i) and (ii). Thus, $L_{22}(s) = [p_{22} x^e(s)]^2 - 1$ satisfies the Nyquist encirclement condition and no RHP pole-zero cancellations occur between $g_{22}(s)$ and $[p_{22} x^e(s)]^2 - 1$.

Once the whole controller $G_\beta$ is designed, the remaining sufficient stability conditions (iii) and (iv) are checked. As a result, the 2x2 subsystem is concluded to be stable. Finally, it is also checked that no additional RHP zeros have been introduced by the compensator.

**SISO subsystem (loop 3): Design of $g_{33}$**

The compensator $g_{33}$ (96) is calculated by using standard QFT loop-shaping for the plant $p_{33}$. Figure 28 shows the nominal loop transfer function with the designed controller and the bounds, which have been calculated from the loop-by-loop specifications [(87) to (90)] and the plant uncertainty.

$$g_{33} = \frac{0.06 (0.014) (0.01)}{(0) (1.2) (1.1)}$$  \hfill (96)
4.6 Results

This Section shows some of the most representative results, illustrating how the methodology can simultaneously handle both loop-by-loop and matrix specifications. The control system is studied in both the frequency domain and the time domain.
Frequency domain analysis

The control system is assessed in the frequency domain in two different ways. First, the QFT robust performance and stability specifications are checked throughout the design procedure, more precisely, at the end of each loop design. The specifications involved are not only the classical QFT robust specifications [(87) to (89)], but also those specifications translated from the matrix requirements [(79) to (86)].

And second, once the whole controller has been designed, the classical multivariable specifications on the sensitivity and the complementary sensitivity matrices, $S(s)$ and $T(s)$ respectively, are analysed. The designed MIMO controller simultaneously fulfils all the set of specifications required in the example, the loop-by-loop and the matrix specifications.

**QFT robust performance and stability specifications**

When completing the QFT design for each system loop, and with the aim of analysing the final results, a plot of the worst (over the uncertainty) closed-loop response magnitude versus the specification is generated at frequencies other than those used for computing the bounds. Note that it does not correspond to the frequency response of just one uncertain case. On the contrary, for each frequency depicted, the frequency response of the worst case within the uncertainty at that particular frequency point is plotted.

The following figures (Figure 30 to 36) show the achievement of the requirements with the above controller design. As mentioned above, the solid-lines represent the worst case at every frequency within the plant uncertainty of the loops, i.e., $L_{ii}(s) = [p_{ii} x^e(s)]^{-1} g_{ii}^\beta(s)$ (i = 1, 2) for the MIMO subsystem (loops 1 and 2), and $L_{33}(s) = p_{33}(s) g_{33}(s)$ for the SISO loop 3. The requirements are presented in dotted-line.

In the case of the MIMO subsystem, it can be found both the classical QFT specifications (Figure 30 and 33) and the loop-by-loop specifications obtained by translation of the classical MIMO specifications on $S(s)$ and $T(s)$ (78), which can be found in Figure 31, 32, 34 and 35. Both the sufficient –green dashed line- and necessary –red dashed line-conditions are plotted.

**Loop 1**

![Fig. 30 Loop 1, classical QFT specifications: (a) Type 1 [equation (87)]; (b) Type 3 [equation (89)]]

---

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Fig. 31 Loop 1, sufficient and necessary specifications translated from the matrix specification for $T(s)$: (a) Type 1A; (b) Type 1B; (c) Type 1C

Fig. 32 Loop 1, sufficient and necessary specifications translated from the matrix specification for $S(s)$: (a) Type 2A; (b) Type 2B
Loop 2

Fig. 33 Loop 2, classical QFT specifications: (a) Type 1 [equation (87)]; (b) Type 3 [equation (89)]

Fig. 34 Loop 2, sufficient and necessary specifications translated from the matrix specification for $T(s)$: (a) Type 1A; (b) Type 1B

Note that the 1C specification corresponding to the second loop of a 2x2 MIMO system, becomes infinite (does not impose any restriction) when the off-diagonal controller is equal to zero, which is the case for the $G_\beta$ controller. This is the reason why no plot is shown for this specification.
Fig. 35 Loop 2, sufficient and necessary specifications translated from the matrix specification for S(s): (a) Type 2A; (b) Type 2B

Loop 3

Fig. 36 Loop 3, classical QFT specifications: (a) Type 1 [equation (87)]; (b) Type 2 [equation (88)]; (c) Type 3 [equation (89)]
**Classical MIMO matrix specifications**

The results for the maximum H-infinity norms are: \( \|S(s)\|_\infty = 2.76 \text{ dB} \); \( \|T(s)\|_\infty = 0.81 \text{ dB} \), which satisfactory fulfil the required specifications. In the following figure, the \( S(s) \) and \( T(s) \) margins (solid line) are plotted versus the frequency for different cases within the uncertainty [Figure 37(a) and Figure 37(b), respectively]. It can be observed that the singular value of the matrix specification (dashed-line) is fulfilled for all the frequencies.

![Graph showing S(s) and T(s) margins](image)

**Fig. 37 Margin for: (a) S(s); (b) T(s)**

**Time domain simulation performance analysis**

The designed control system is checked through time domain simulations for different cases within the uncertainty. The simulator takes into account step perturbations (worst case in rate) at plant input for the different loops (see Table IX). Additionally, the control signal along the three axes is limited not only in value but also in rate (see Table IX).

<table>
<thead>
<tr>
<th>Perturbation at plant input loop 1</th>
<th>Perturbation at plant input loop 2</th>
<th>Perturbation at plant input loop 3</th>
<th>Actuator rate limit</th>
<th>Actuator saturation limit</th>
</tr>
</thead>
<tbody>
<tr>
<td>50 N step at ( t = 4000 ) sec</td>
<td>50 N step at ( t = 8000 ) sec</td>
<td>50 N at ( t = 6000 ) sec</td>
<td>( \pm 75 ) N/sec</td>
<td>( \pm 100 ) N</td>
</tr>
</tbody>
</table>

In the following set of figures (Figure 38 to 40), the results for position measures are presented for each axis. The detailed figures show the effect of uncertainty on the time domain response. Nine representative cases delimiting the response for the rest of plant cases are shown. They correspond to the combination of the minimum, mean and maximum value of the mass \( m \) and the mean orbit rate \( \omega_0 \).
Fig. 38 Time domain results for position x: (a) General view; (b) Detail

Fig. 39 Time domain results for position y: (a) General view; (b) Detail

Fig. 40 Time domain results for position z: (a) General view; (b) Detail

The control signals (actuator) corresponding to the above performance results are presented in Figure 41 to 45 for the three axes. The values required stay well within the allowed range of actuation –Table IX-, avoiding any problem of saturation.
Fig. 41 Time domain results: Actuator X-axis

Fig. 42 Time domain results for Actuator X-axis: (a) Detail 1; (b) Detail 2

Fig. 43 Time domain results for Actuator Y-axis
5.0 CONCLUSIONS

Since the very first ideas suggested by Horowitz in 1959 until now, the Quantitative Feedback Theory (QFT) has been successfully applied to many control systems: linear and non-linear, stable and unstable, SISO and MIMO, minimum and non-minimum phase, with time-delay, with lumped and distributed parameters, multi-loop, etc [1]. The method searches for the controller that guarantees the achievement of the required performance specifications for every plant within the existing model uncertainty. QFT highlights the trade-off (quantification) among the simplicity of the controller structure, the minimization of the ‘cost of feedback’, the quantified model uncertainty and the achievement of the desired performance specifications at every frequency of interest.

The first part of the paper summarized a methodology to design sequential non-diagonal QFT controllers for multi-input-multi-output MIMO systems with uncertainty. The second part demonstrated the feasibility of that methodology to control Spacecraft, in particular the position and attitude control of a 6x6 MIMO Darwin-type spacecraft with large flexible appendages. The third part of the paper introduced an example of spacecraft flying in formation, and proved how the reformulation of the new MIMO QFT methodology (see [8] for details) can handle within the QFT framework both classical QFT specifications and requirements issued from matrix specifications. The resulting design fulfils the requirements and is able to
deal with potential problems arising in the decoupling part of the procedure, such as RHP zeros, poles on the imaginary axis, required RHP transmission zeros in the compensator, uncertainty, etc.

ACKNOWLEDGEMENT

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REFERENCES


[5]. Garcia-Sanz M. and Eguinoa I., (2005), "Improved non-diagonal MIMO QFT design technique considering non-minimum phase aspects," 7th International Symposium on Quantitative Feedback Theory QFT and Robust Frequency Domain Methods, Lawrence, Kansas, USA.


[8]. Garcia-Sanz M., Eguinoa I. and Bennani S. “Non-diagonal MIMO QFT controller design reformulation”. International Journal of Robust and Non-Linear Control, Wiley. Accepted for publication, June 2008.


